

Lecture 11

# Integration

Quadrature and Monte Carlo methods

**CS328 - Numerical Methods for  
Visual Computing and Machine Learning**

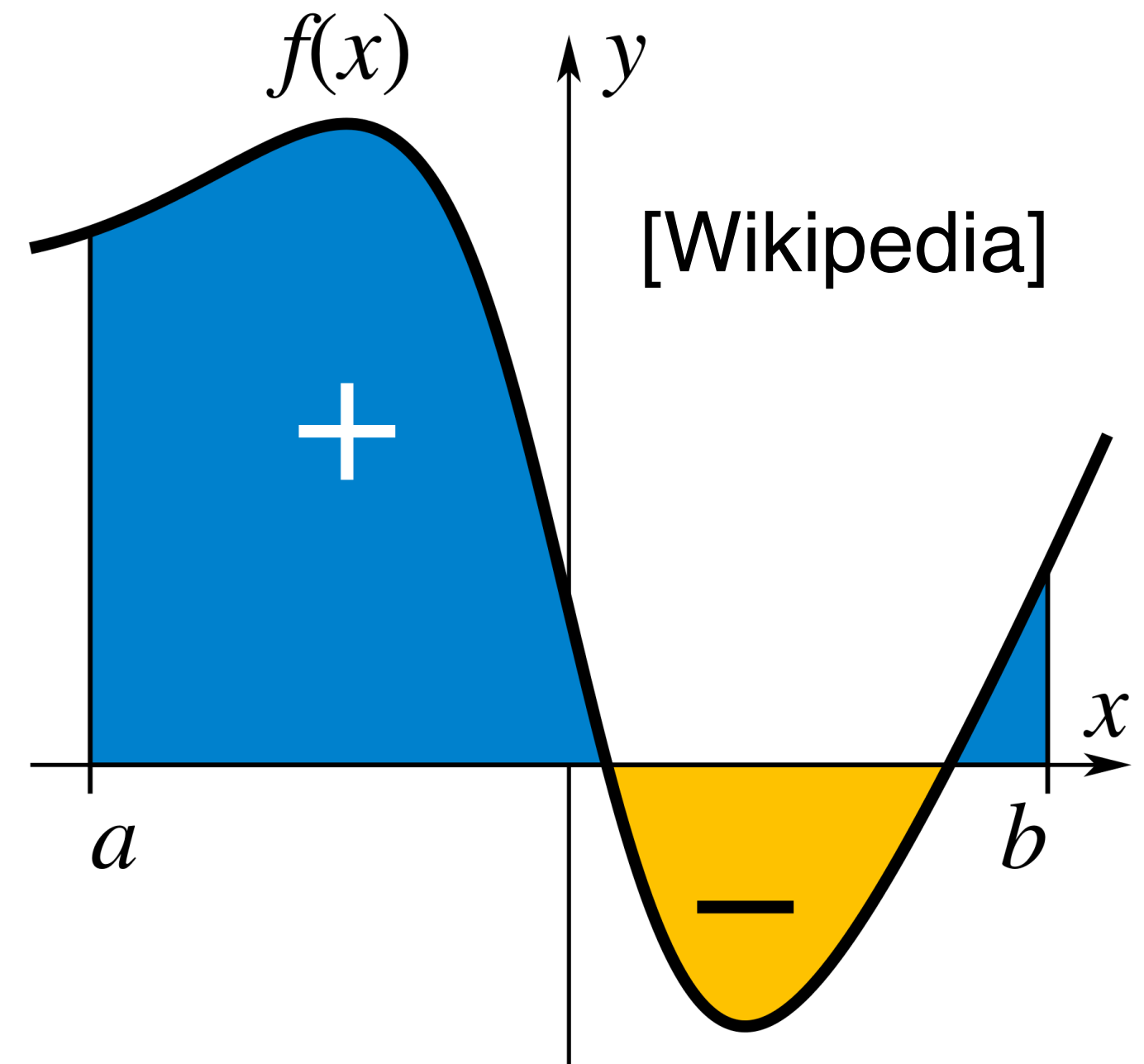
Prof. Wenzel Jakob

# Definite integrals

$$\int_a^b f(x) dx$$

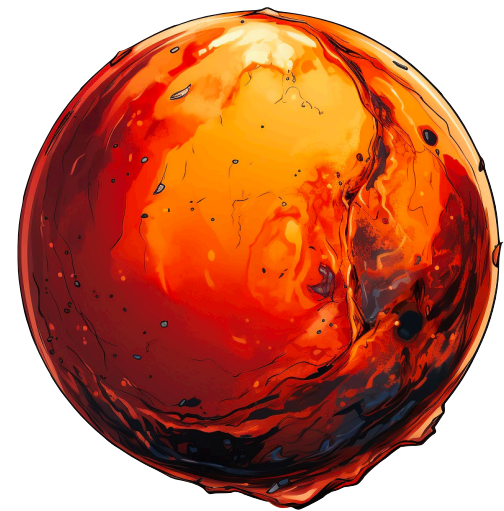
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# Why do integrals arise?

.. for many reasons, but here are two examples of physical processes



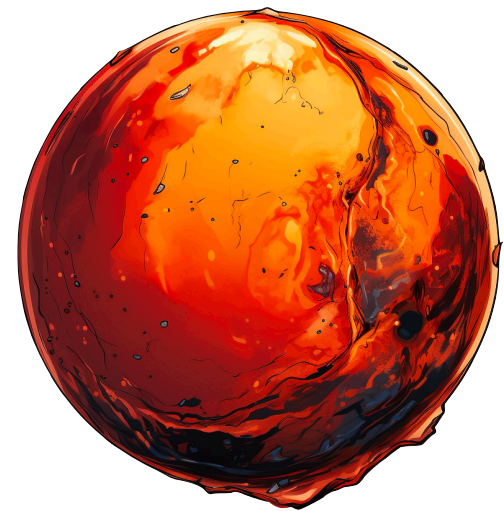
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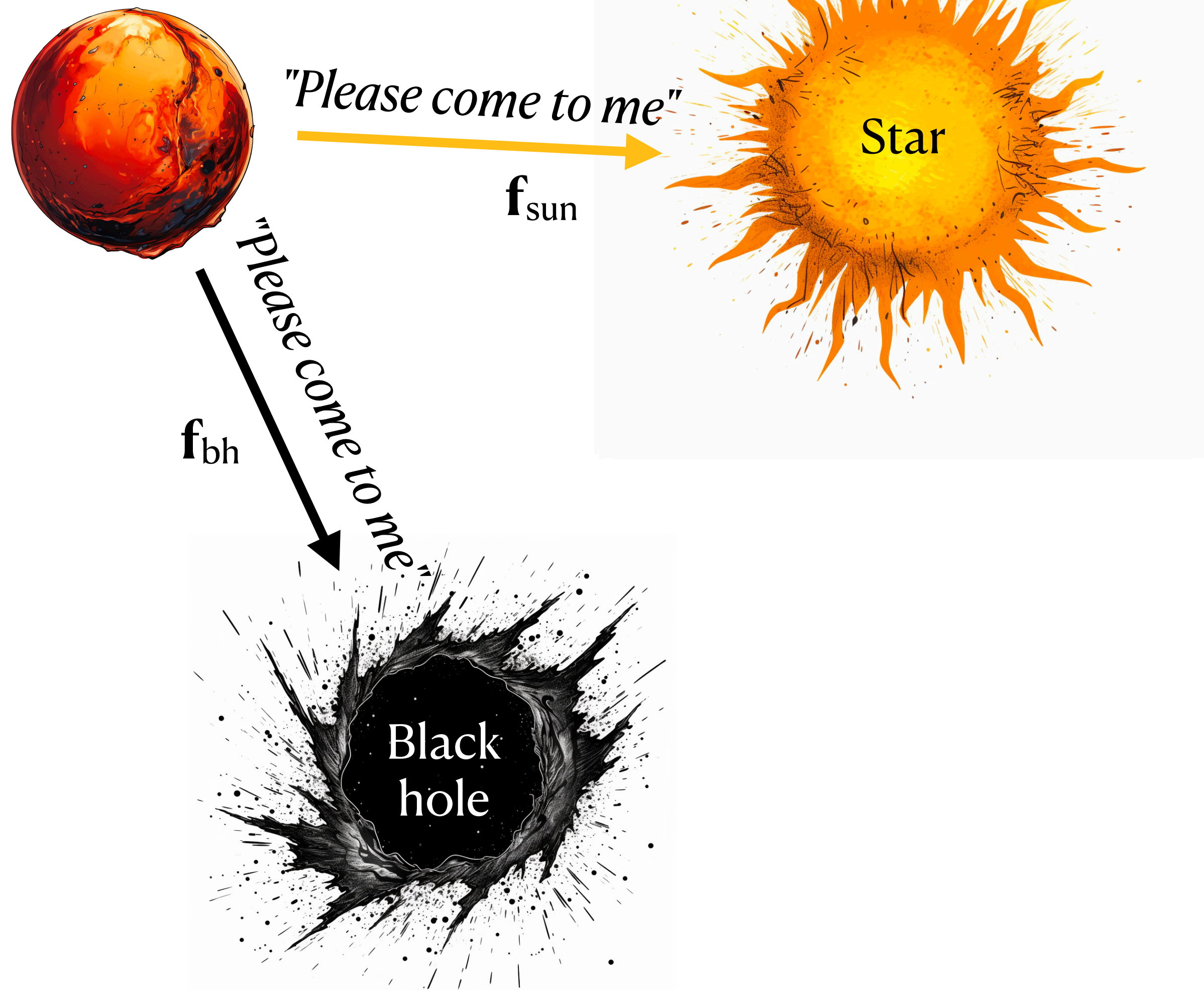
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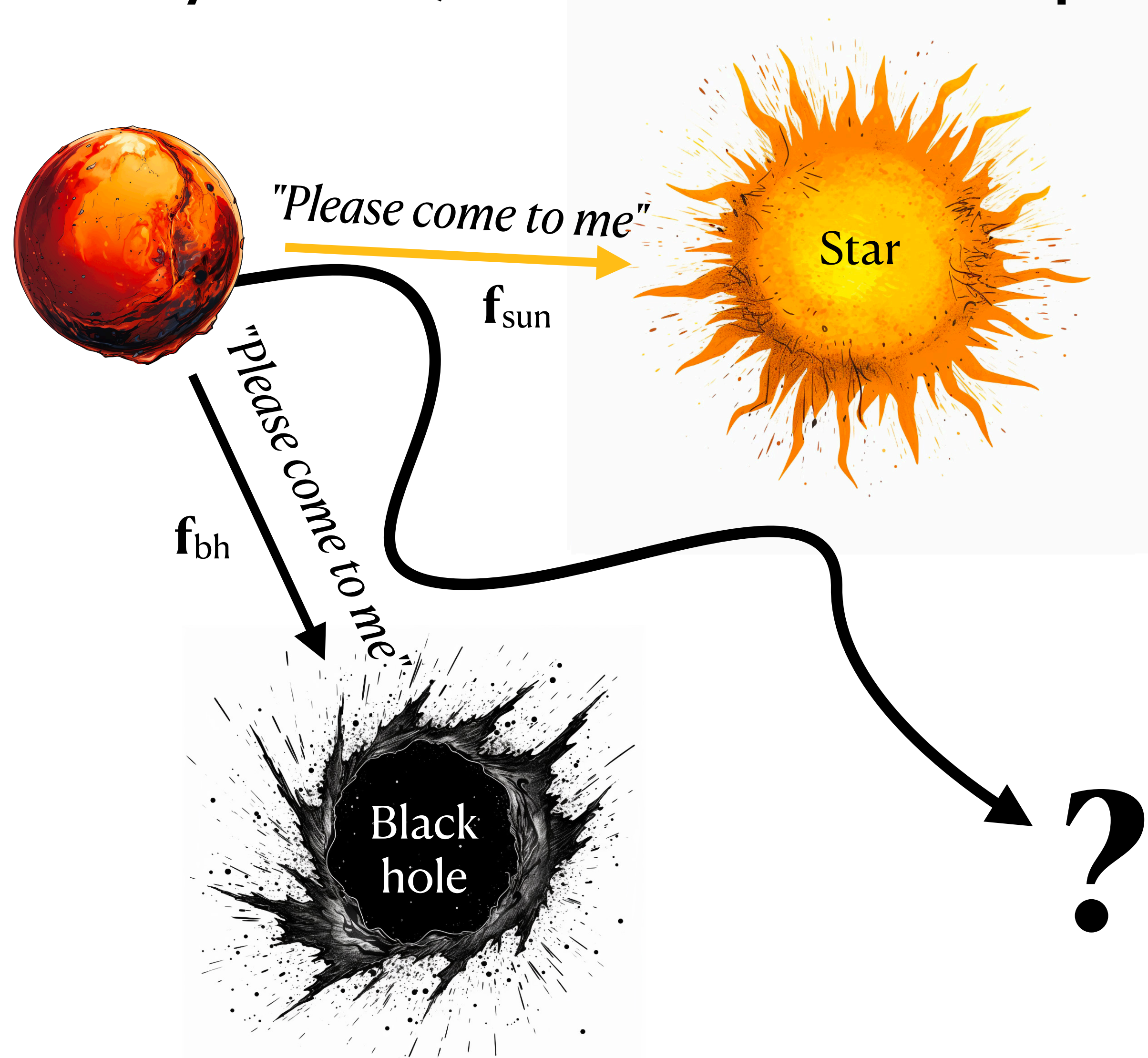
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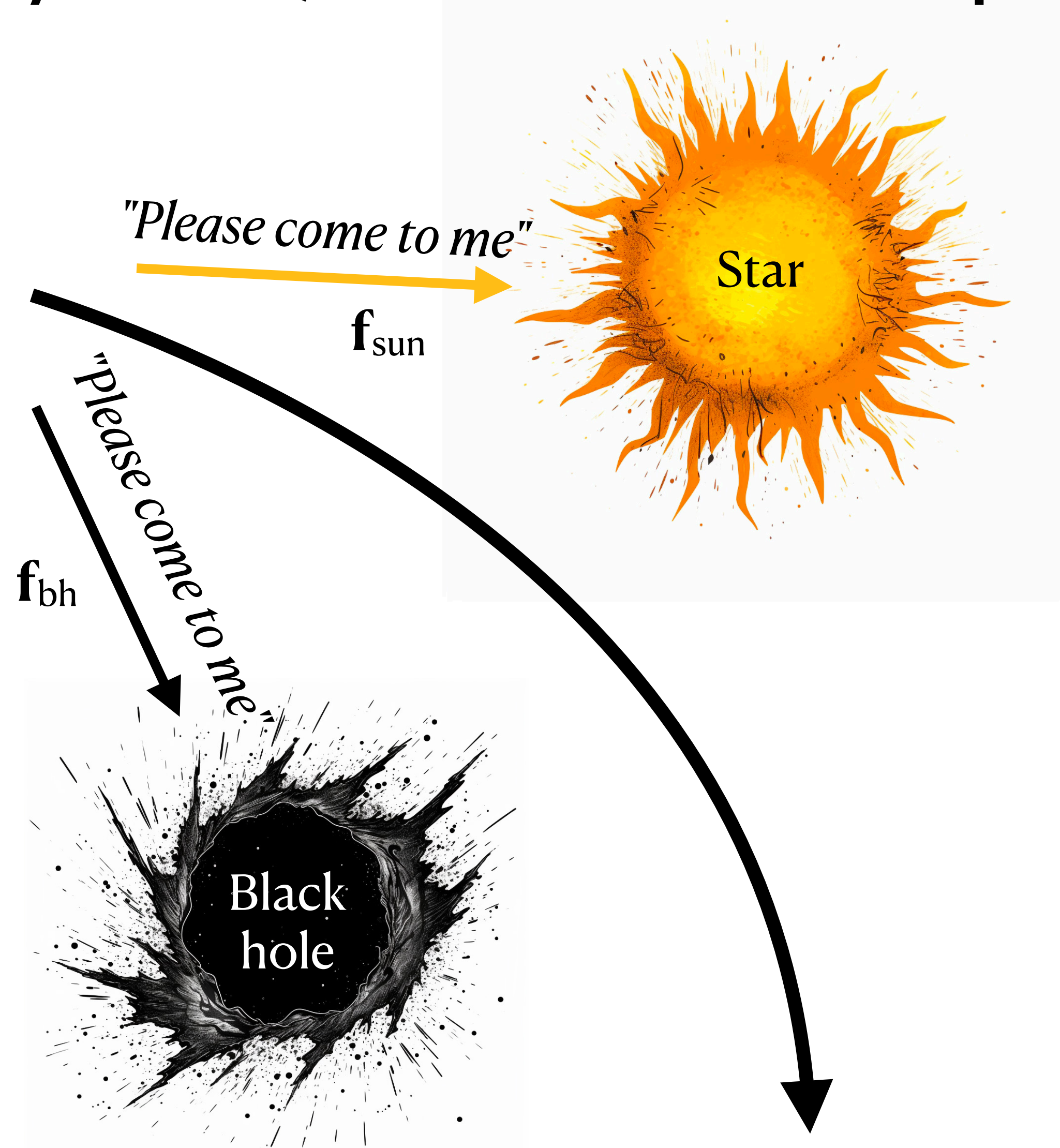
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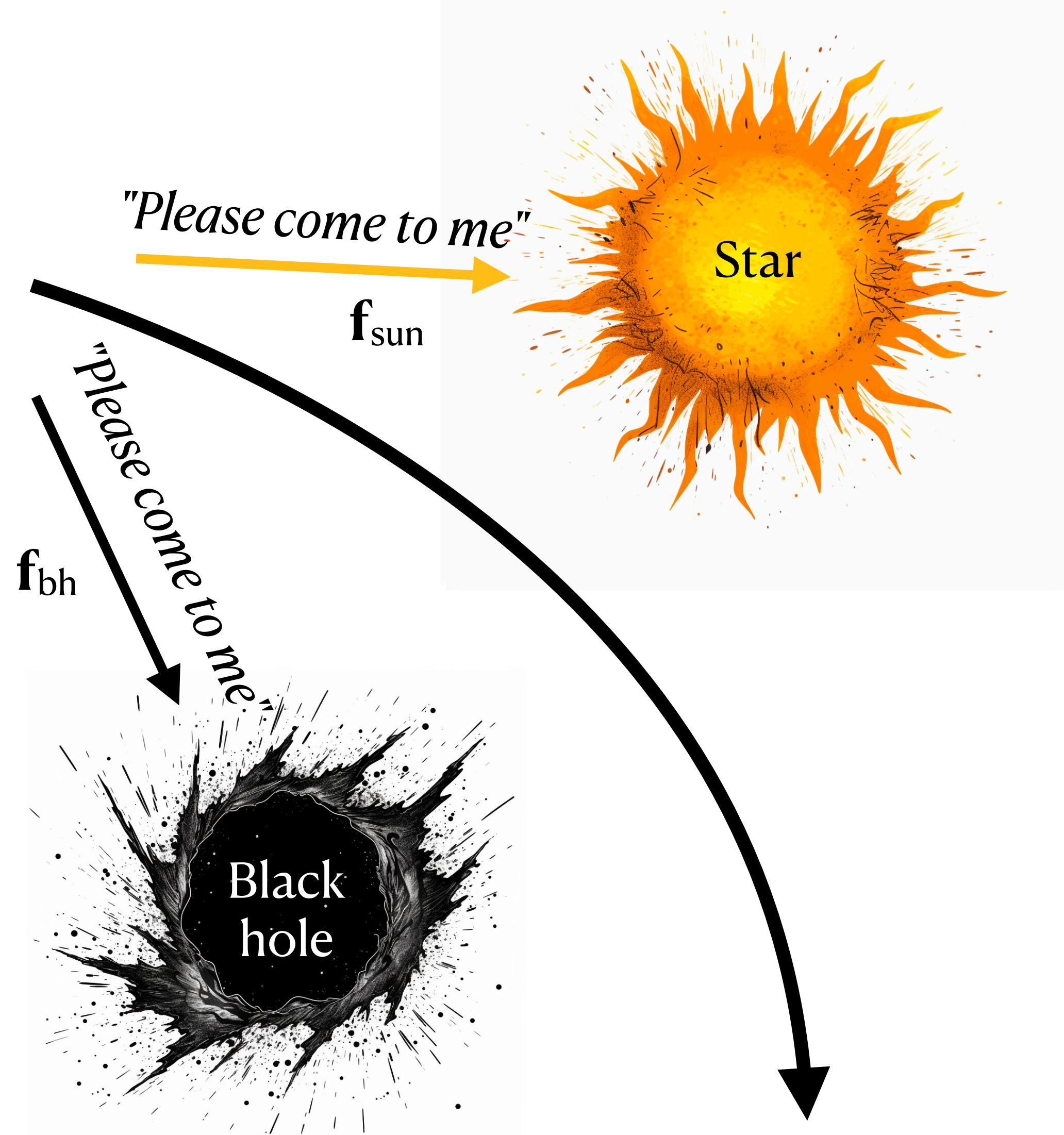
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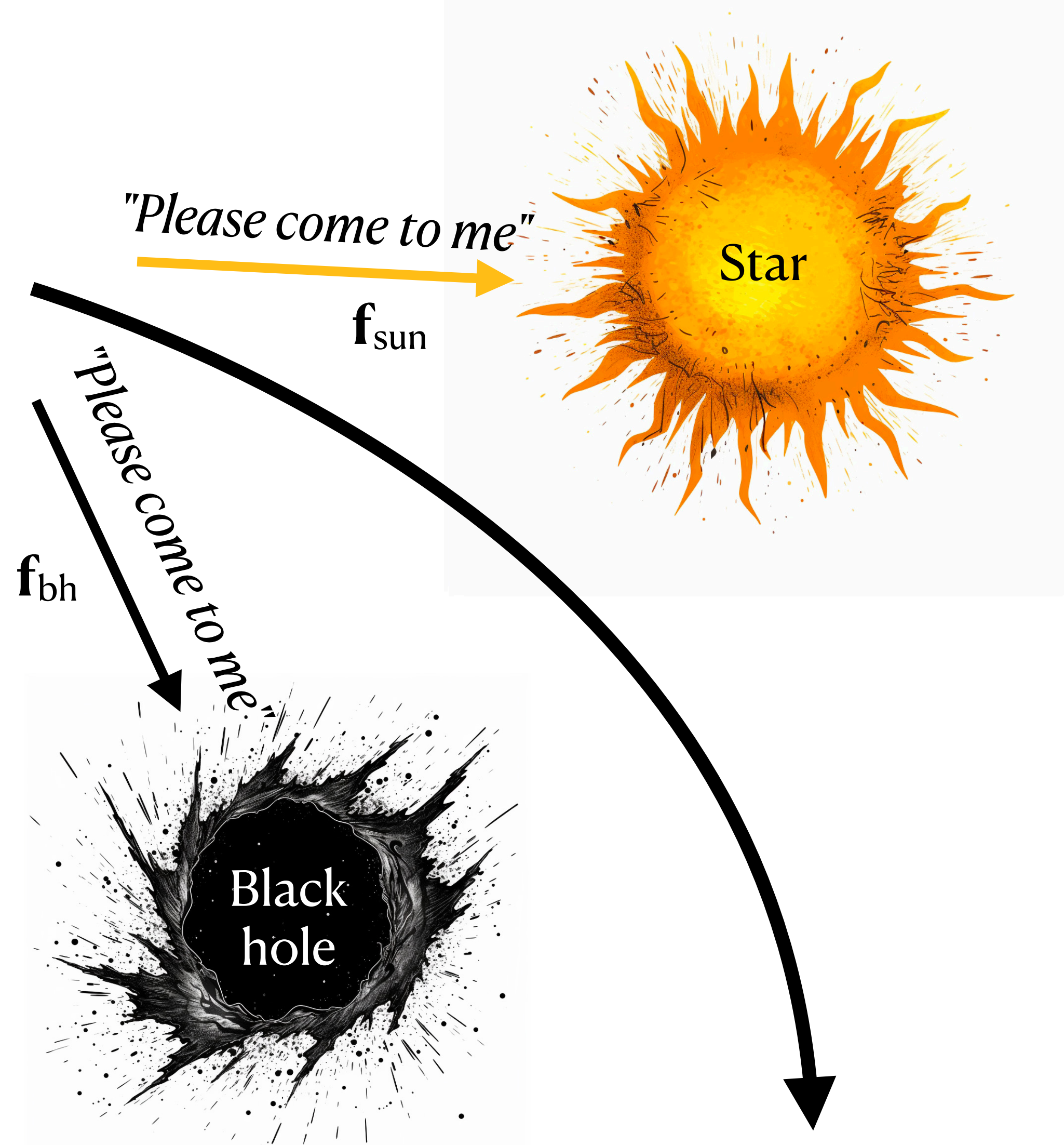


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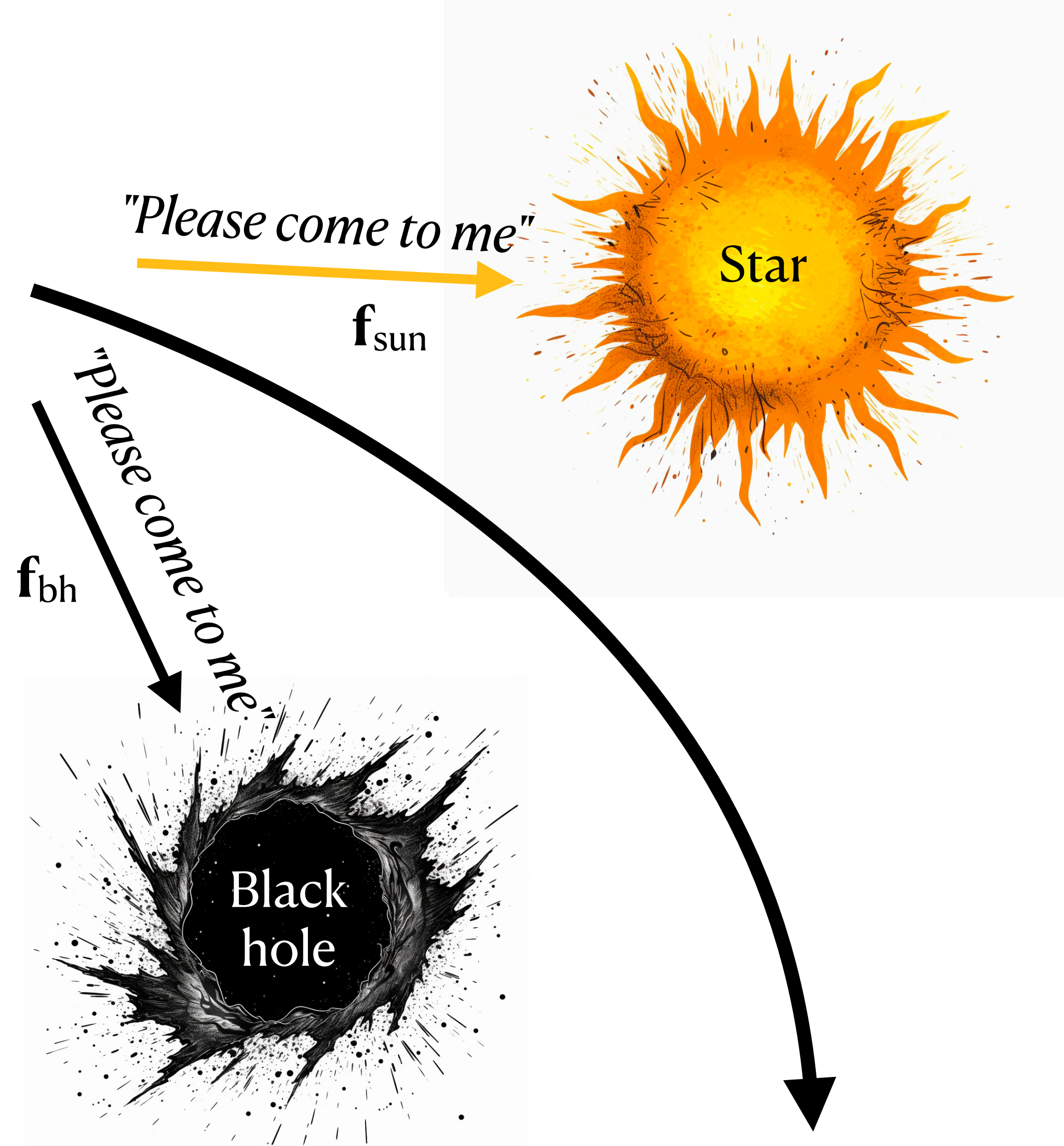
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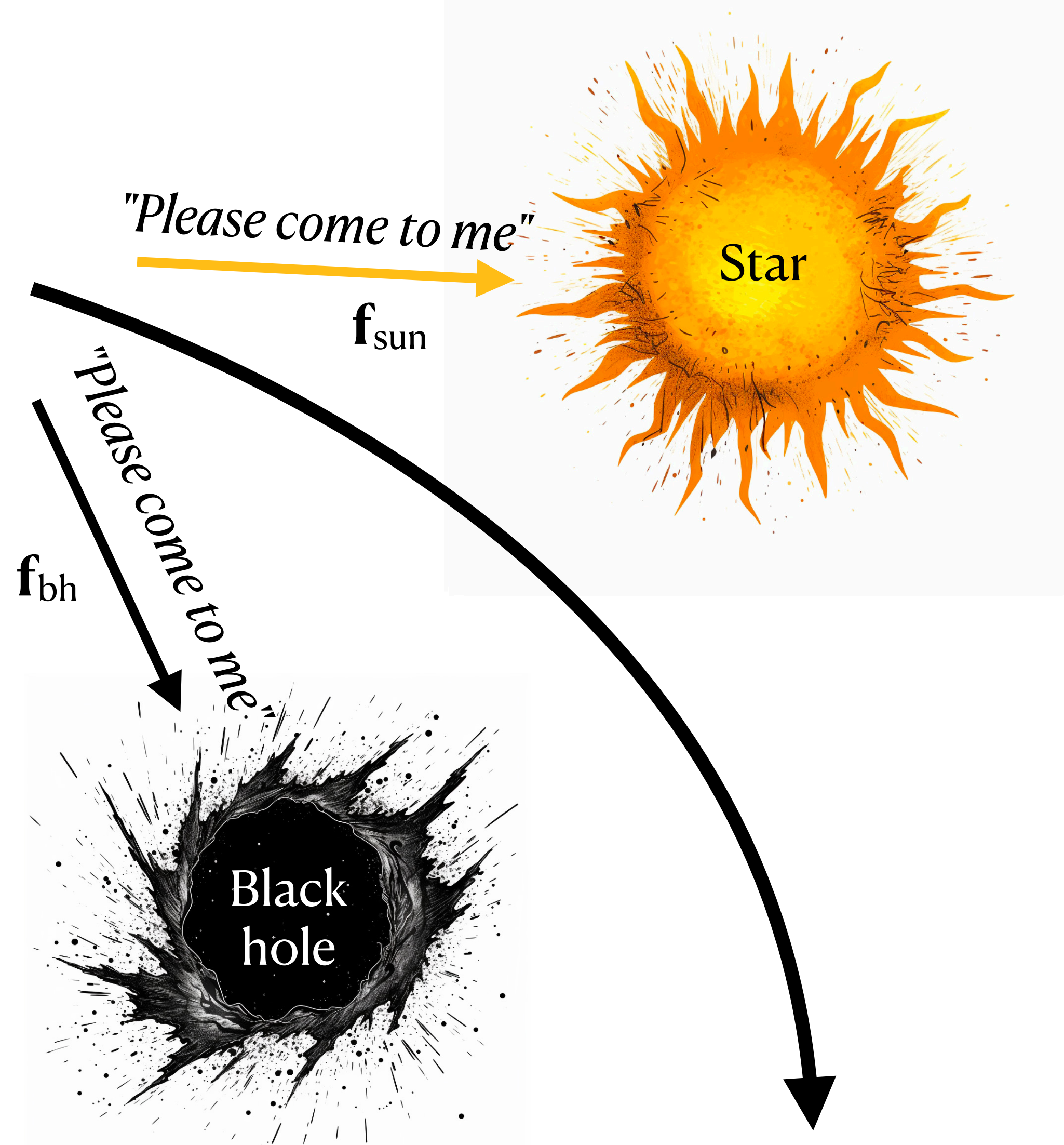
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Mass of the planet

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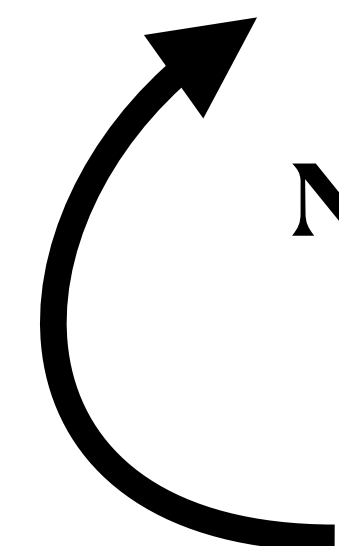
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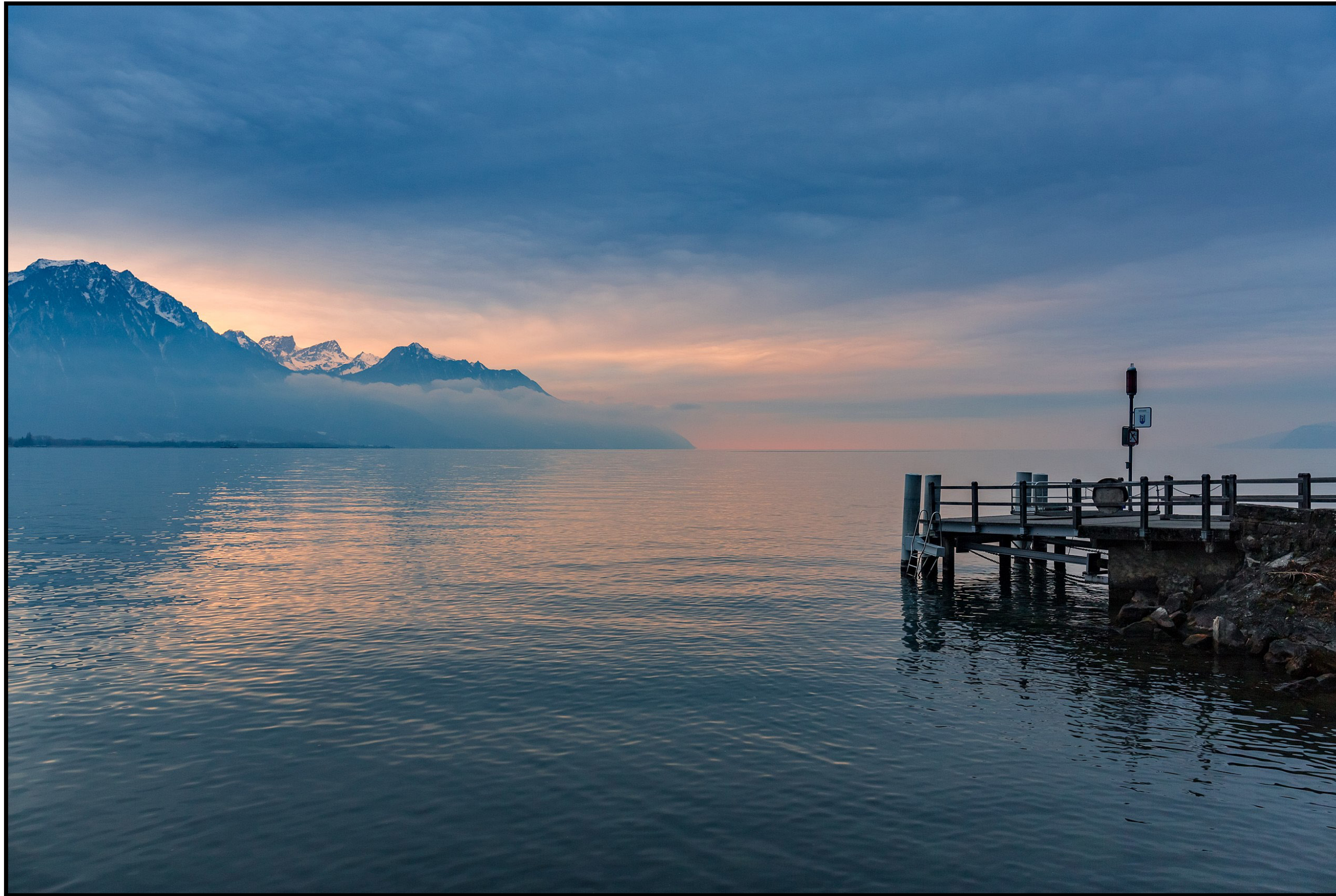
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Physical equation models the evolution of the world as a *time average*.

This average is not discrete but has *infinite resolution*, hence an integral must be used to simulate physics.

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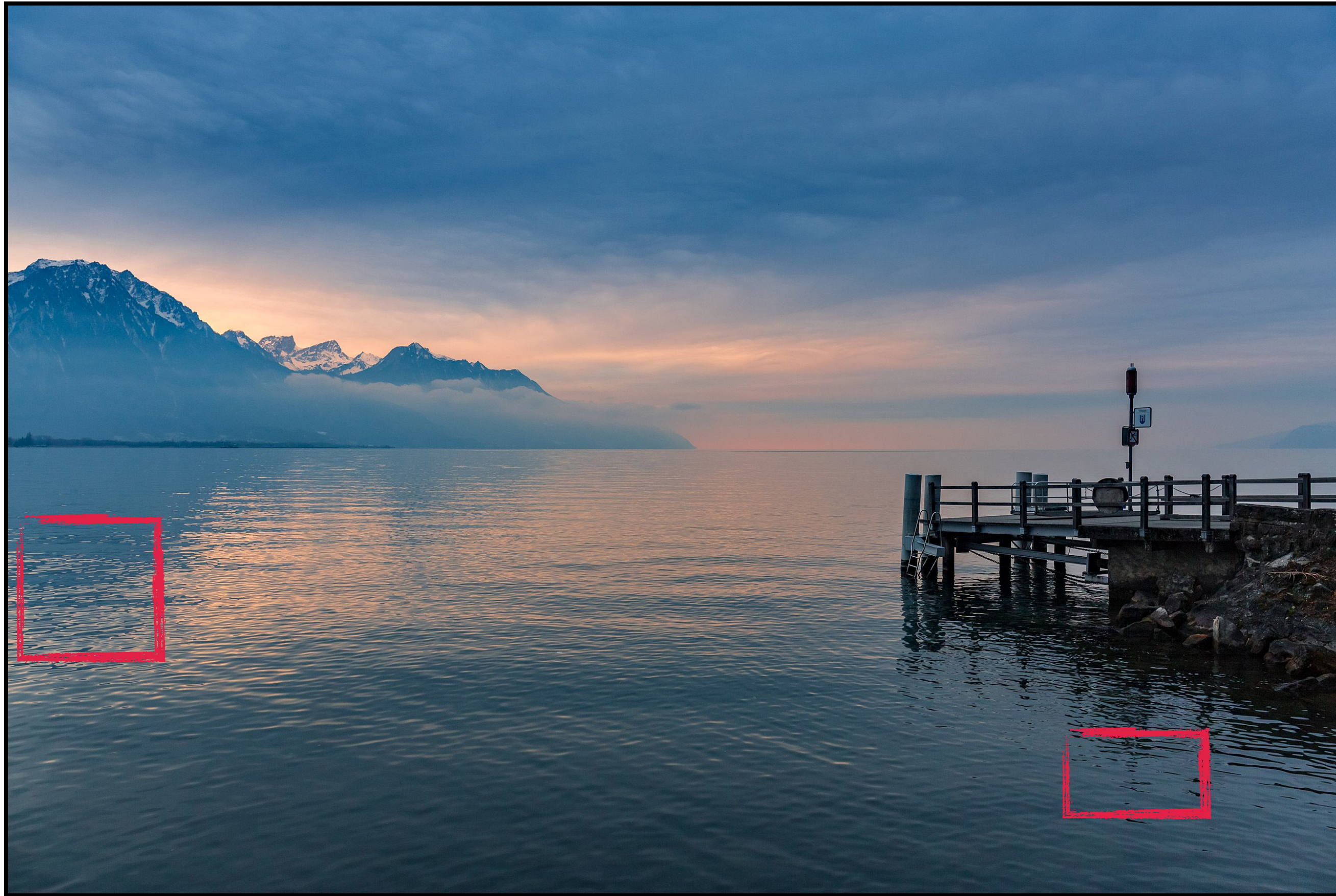
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[Lake Geneva from Chillon Castle, [Wikimedia commons](#)]

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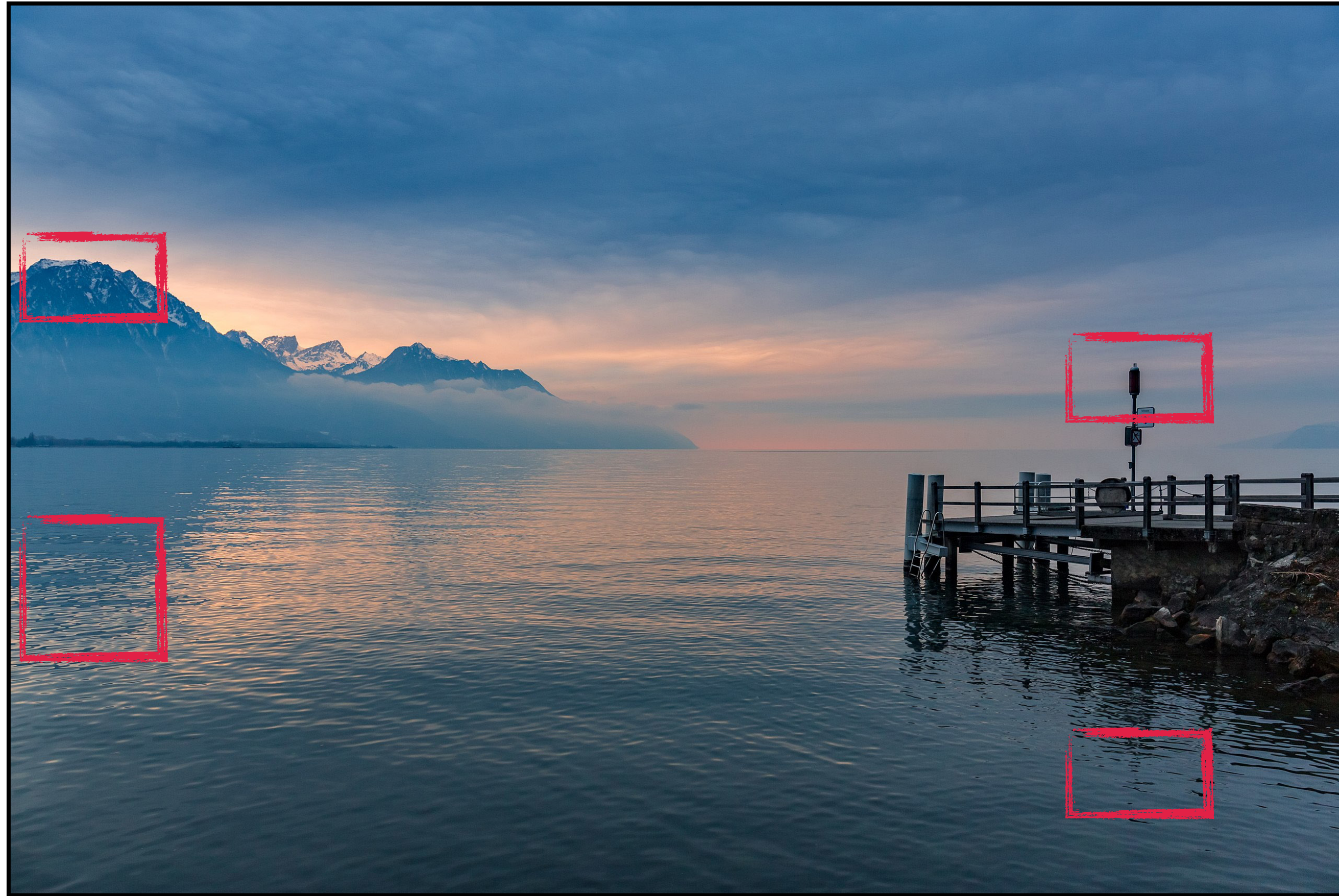
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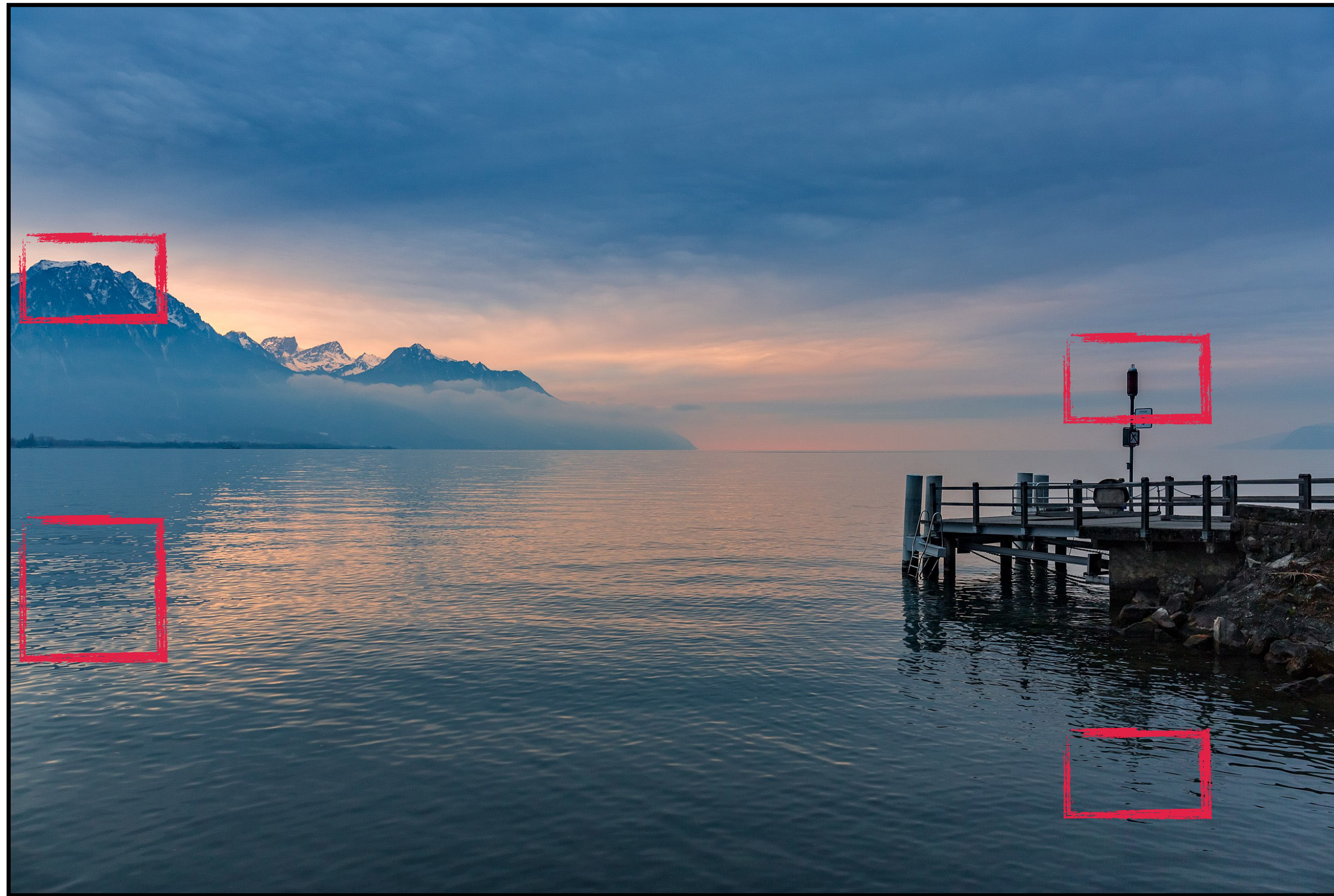
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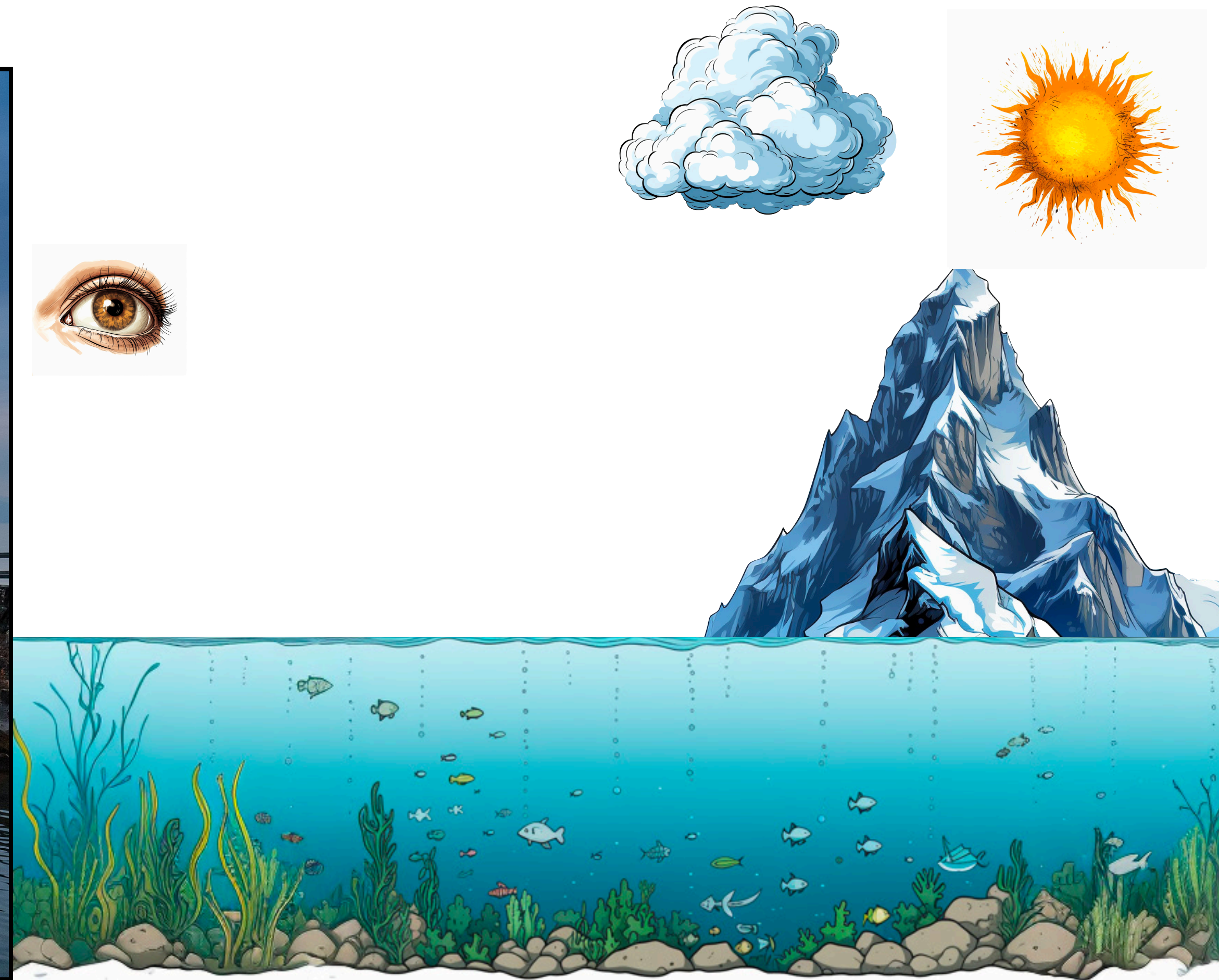
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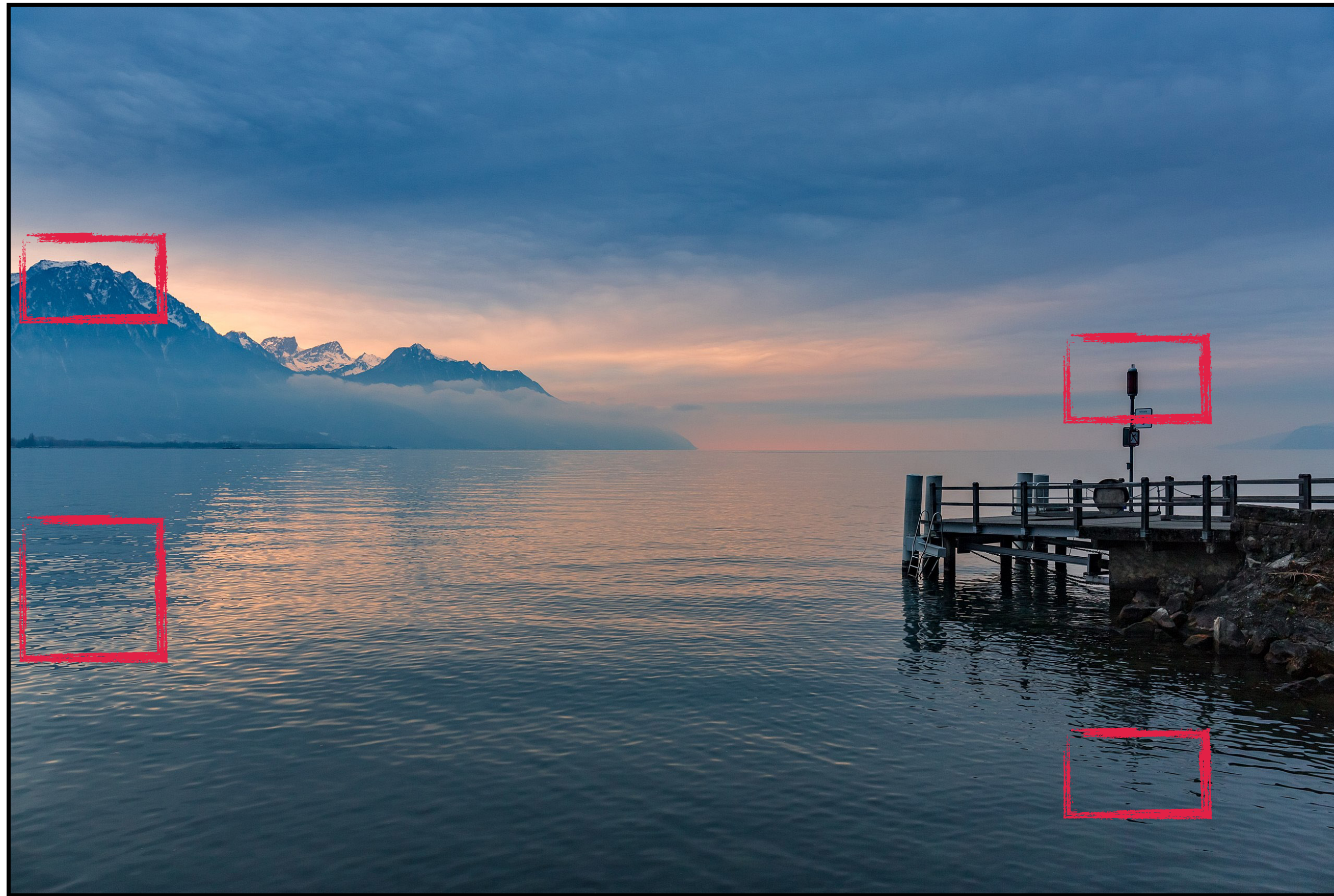


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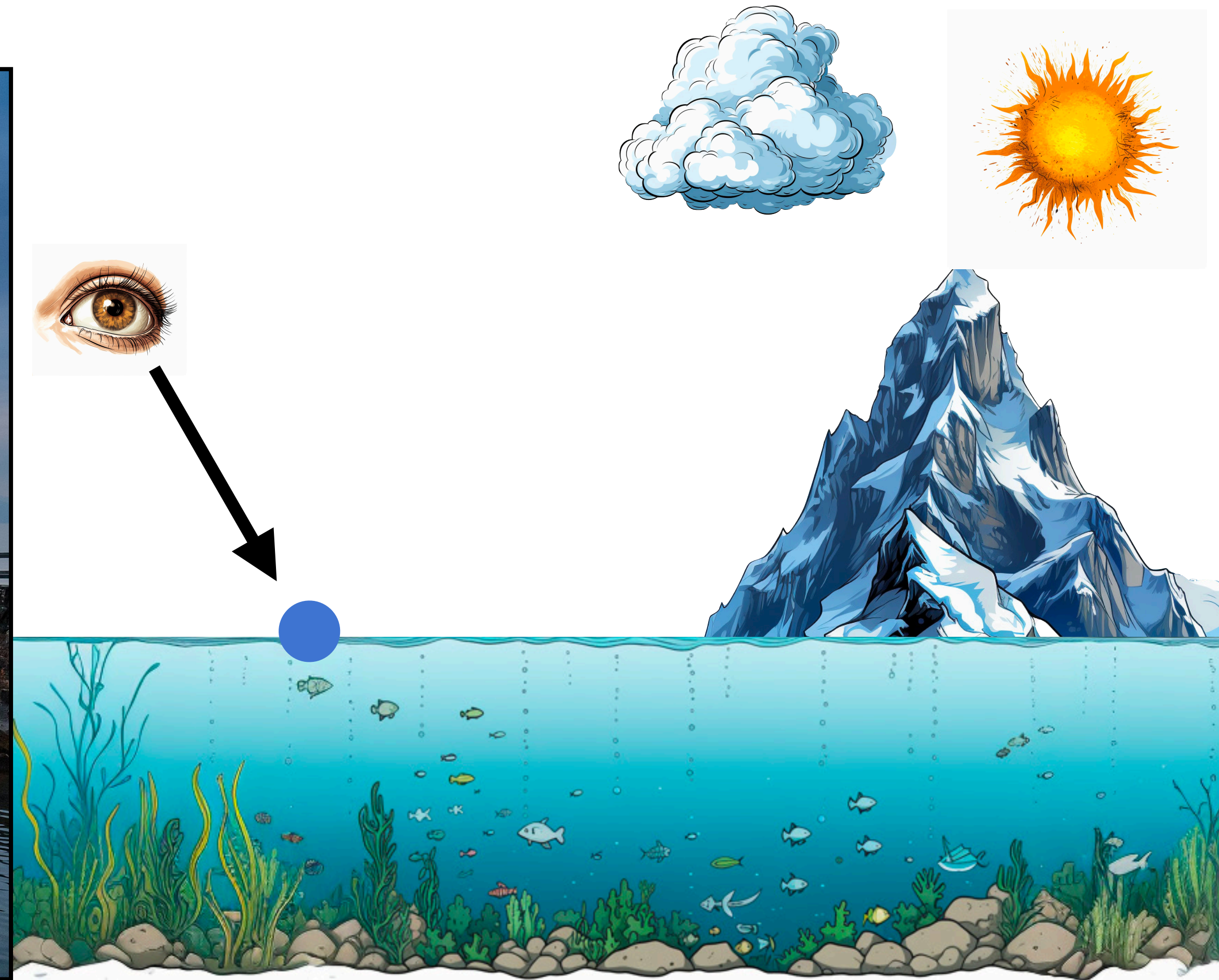


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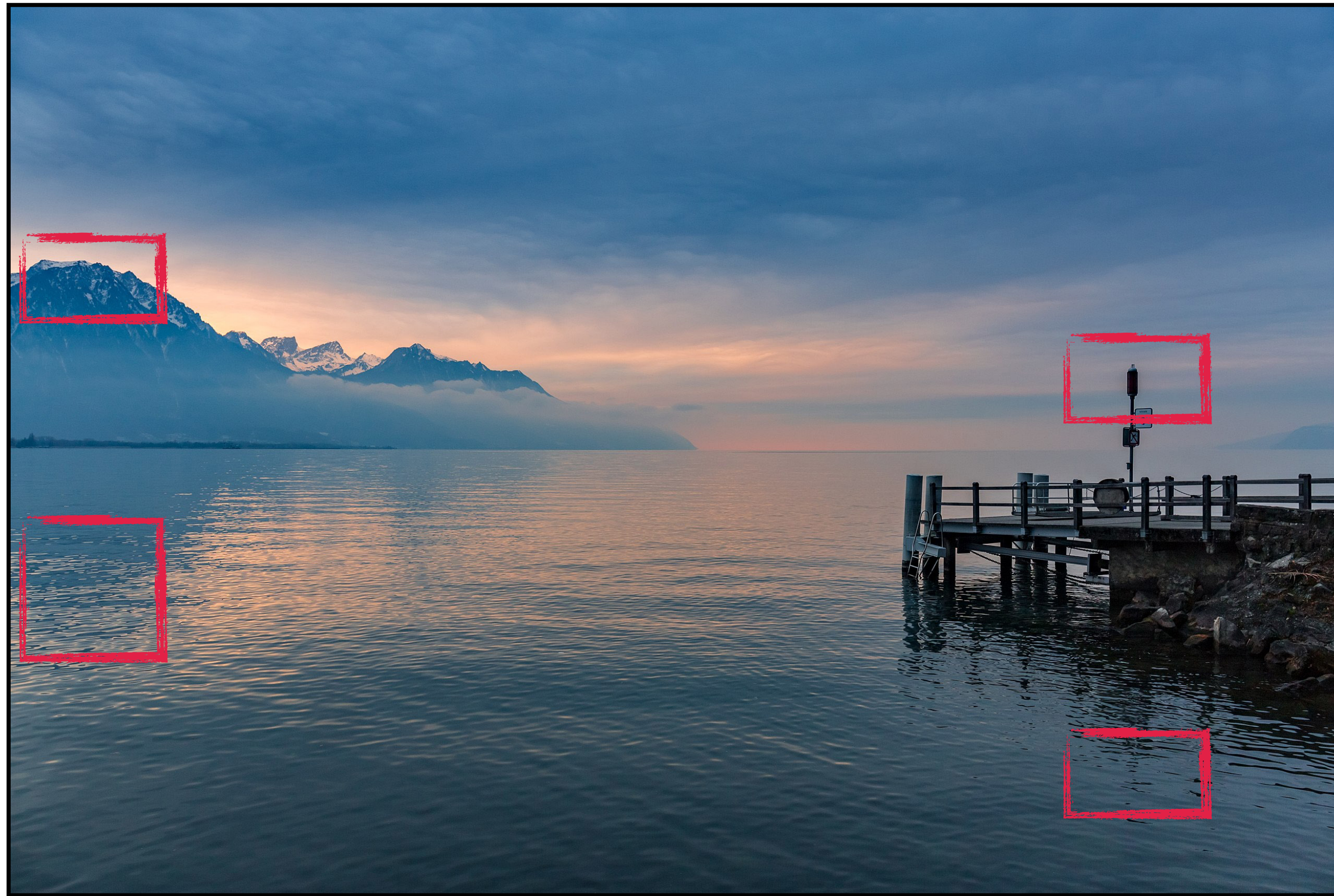


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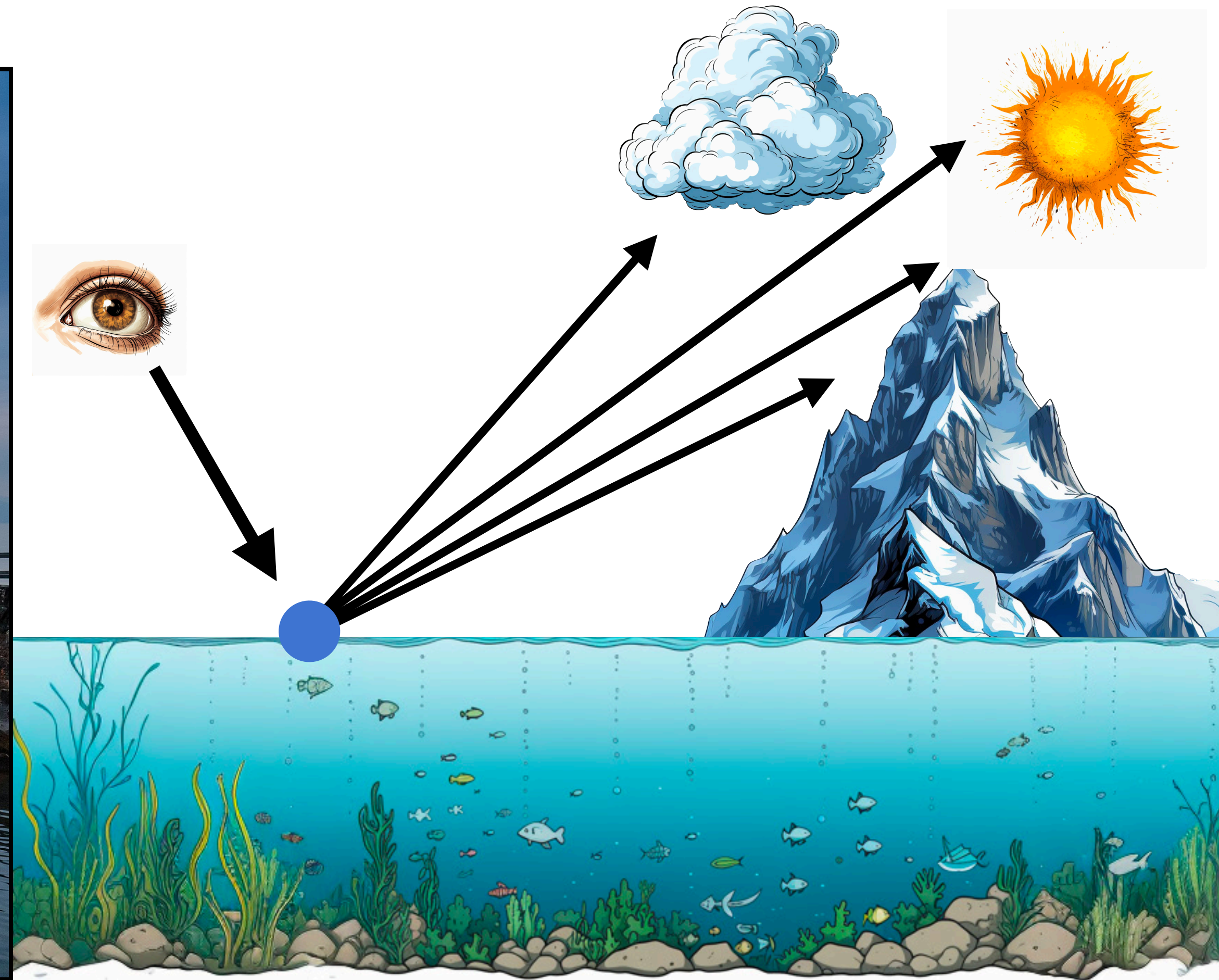


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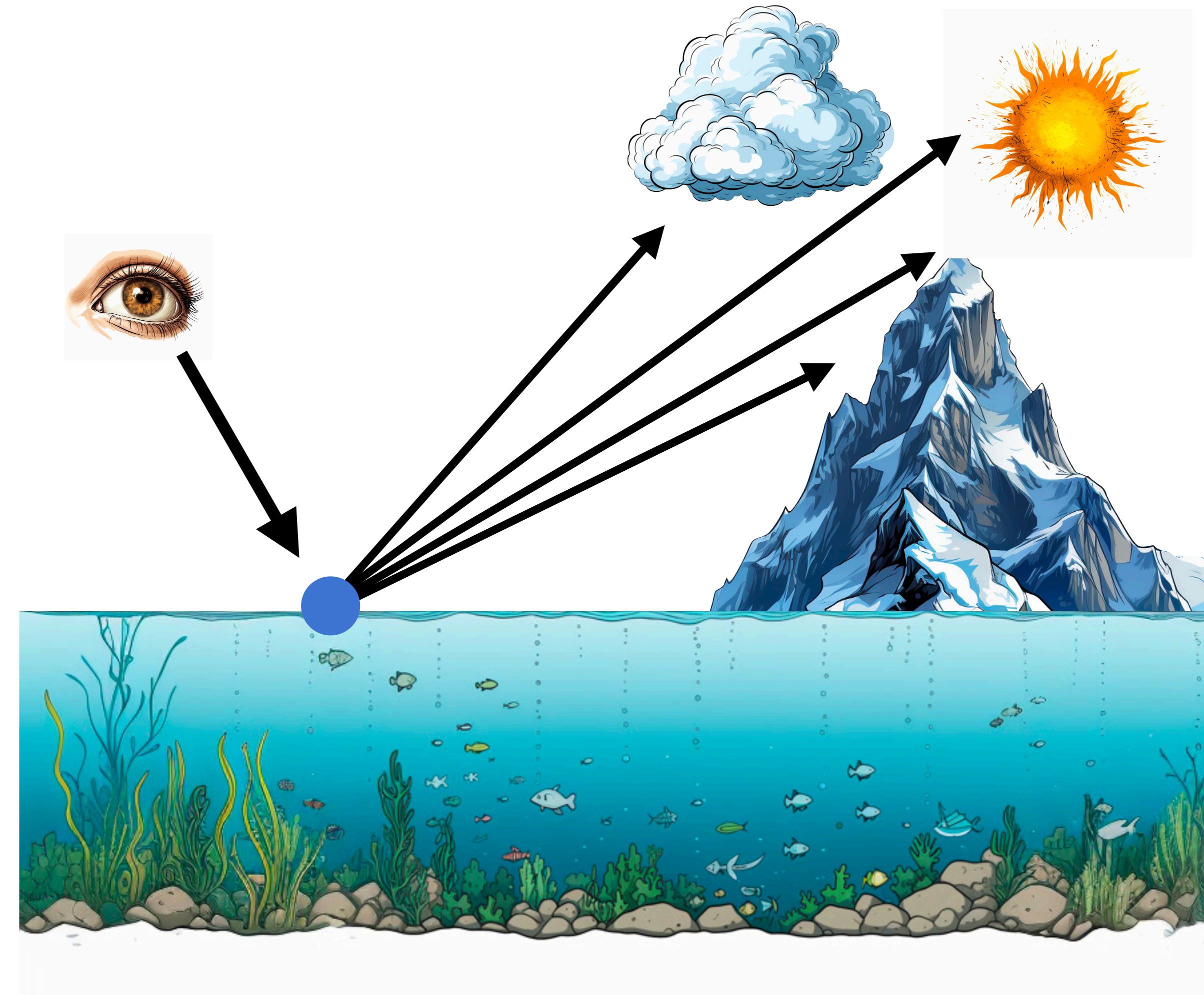


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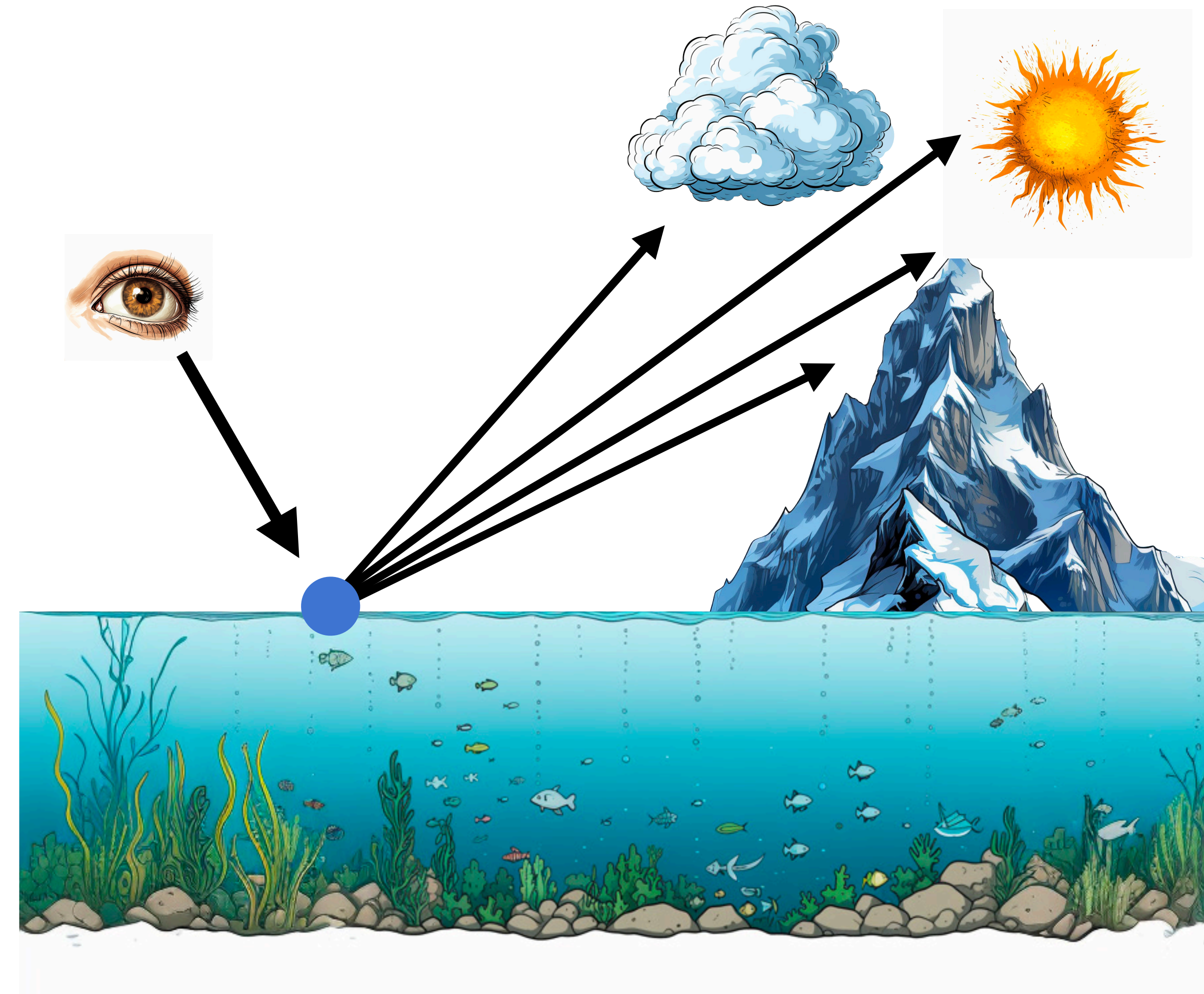
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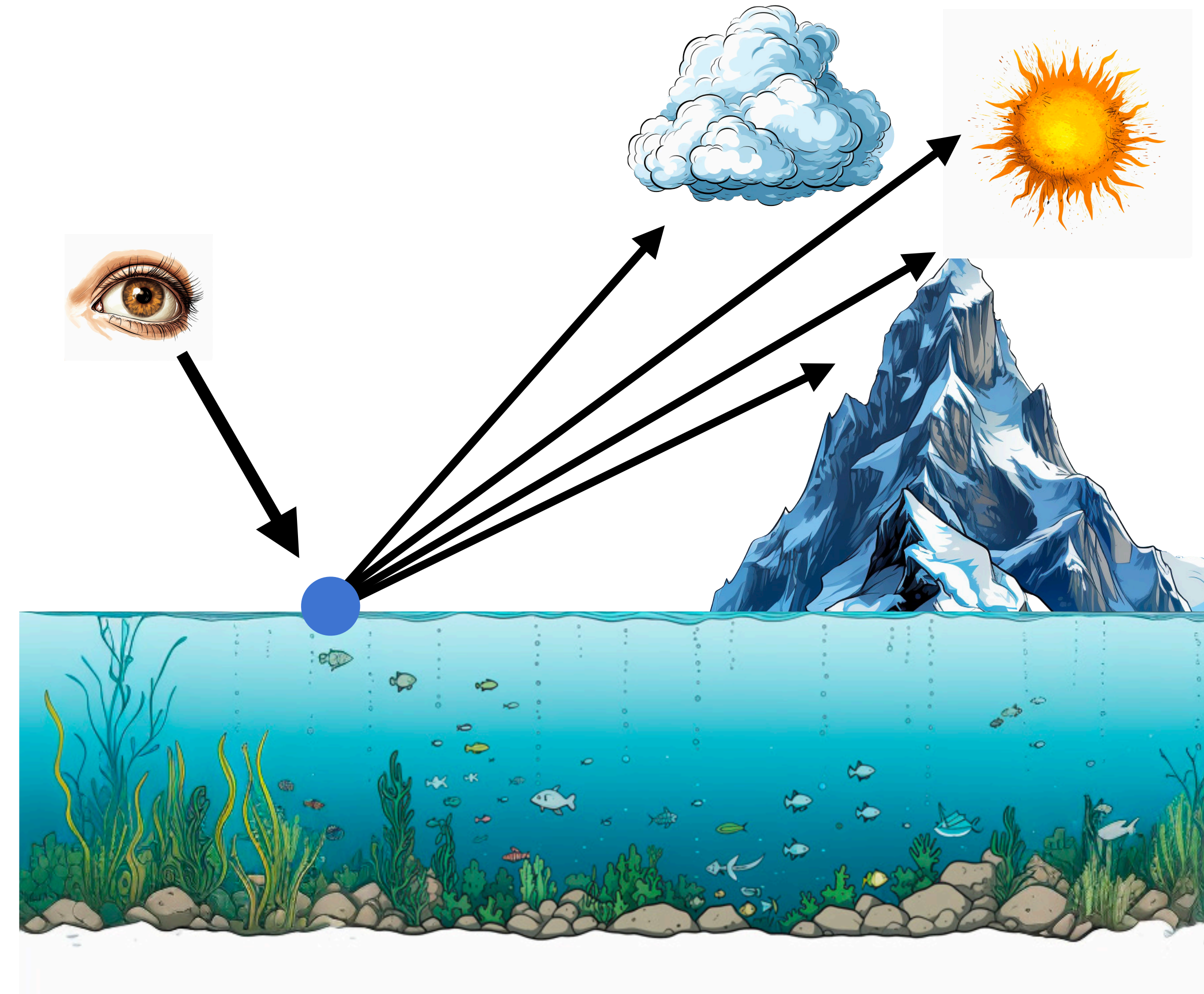
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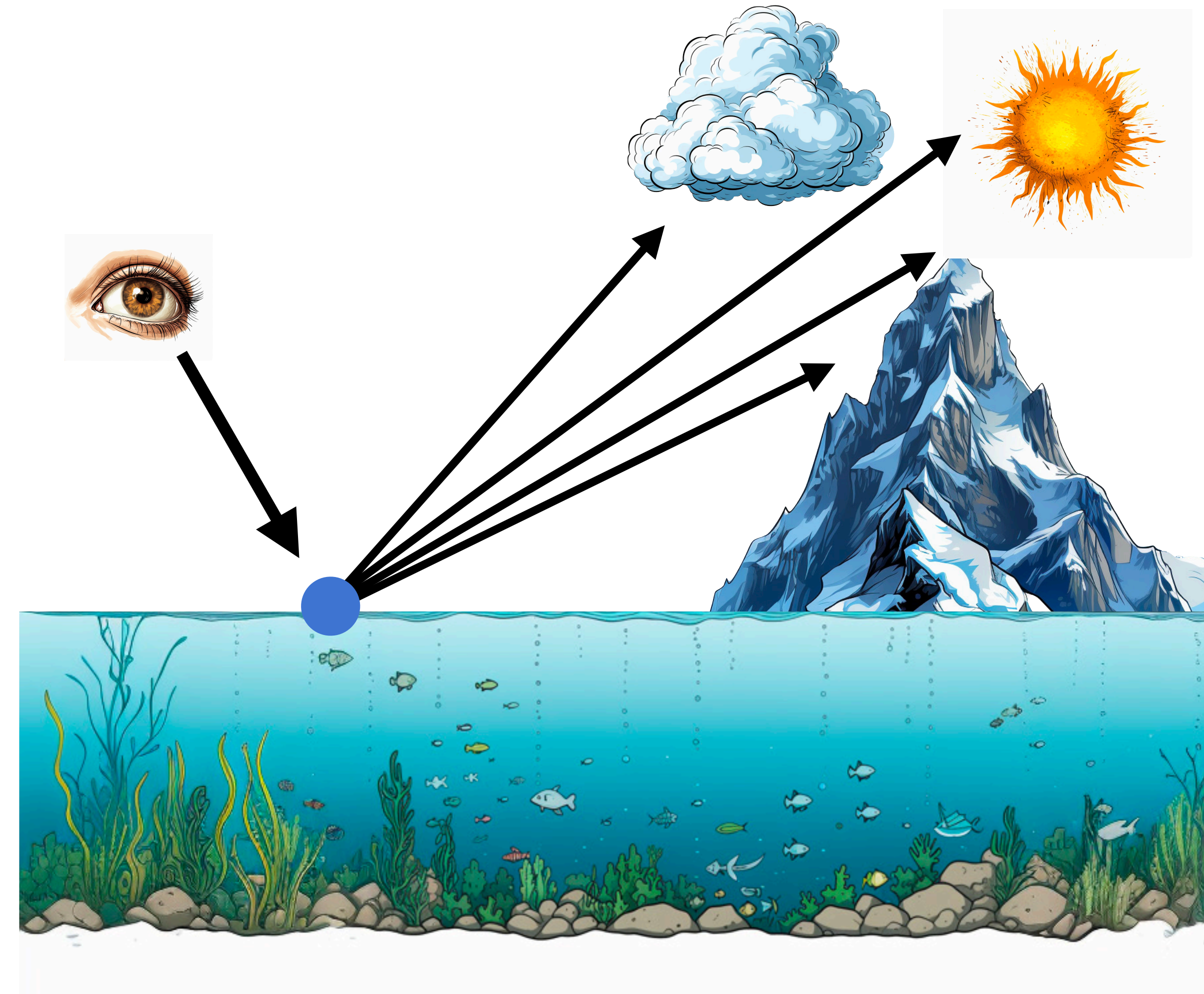
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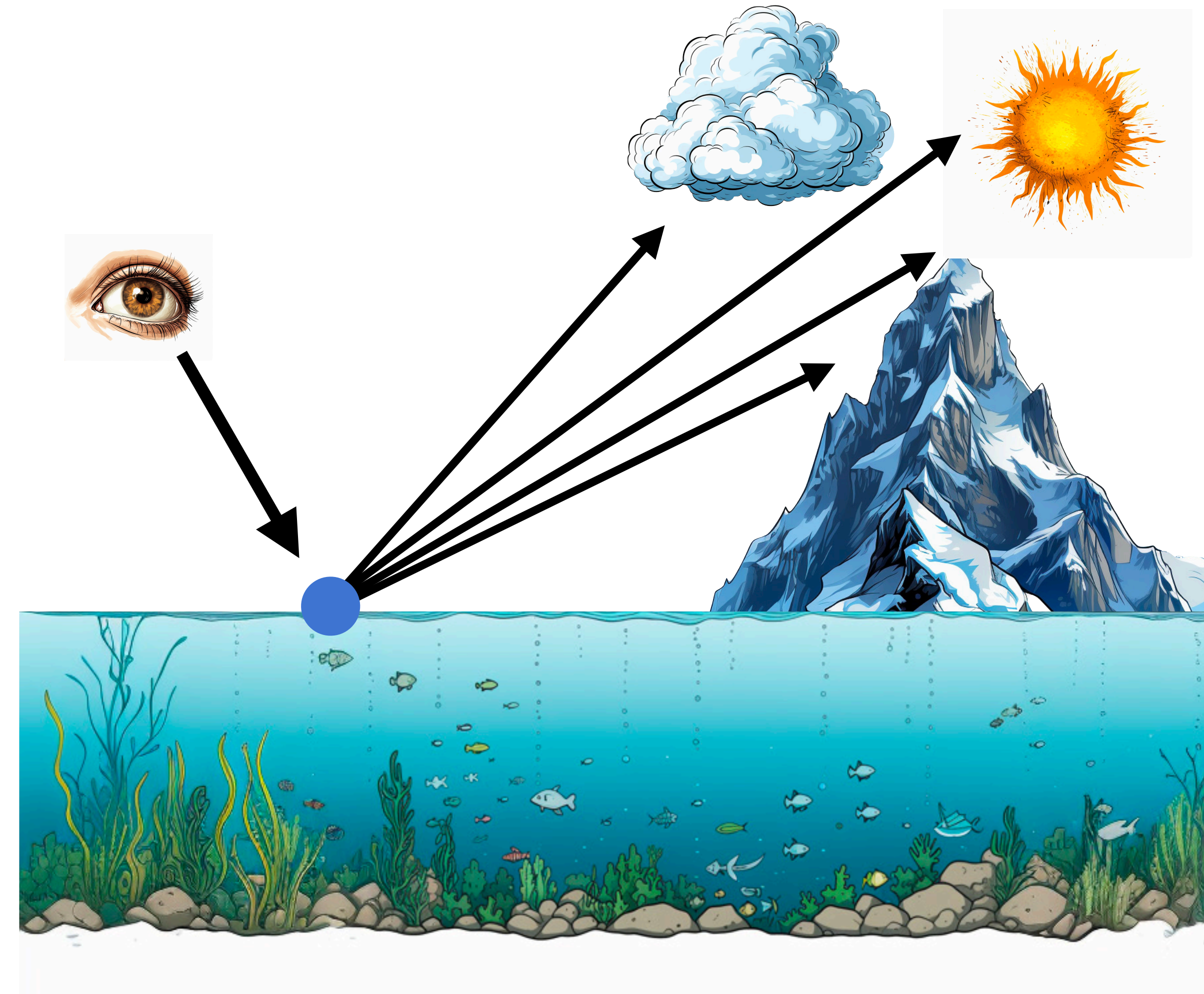
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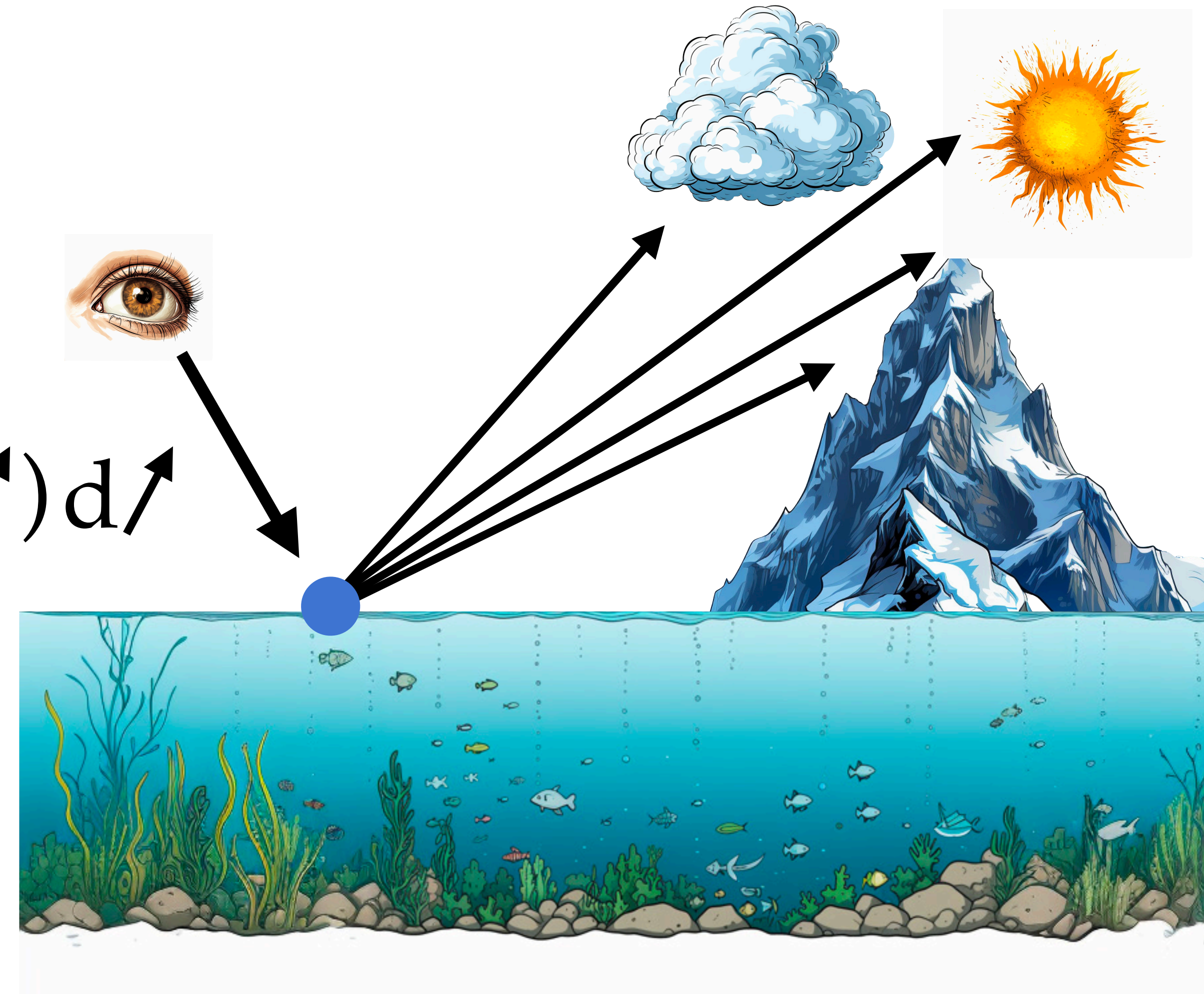
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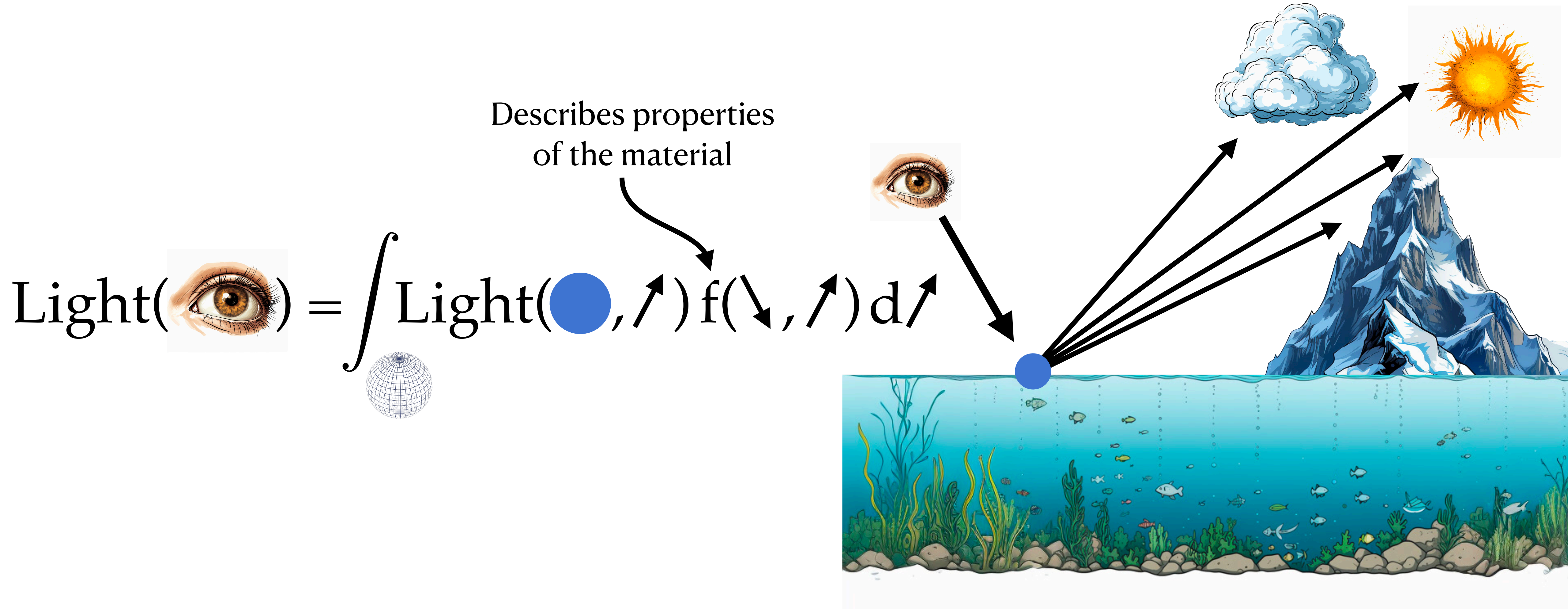
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$$\text{Light}(\text{eye}) = \int_{\text{globe}} \text{Light}(\text{blue dot}, \uparrow) f(\searrow, \uparrow) d\uparrow$$



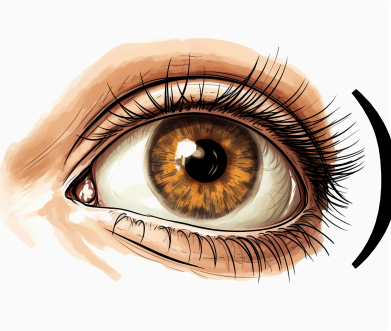
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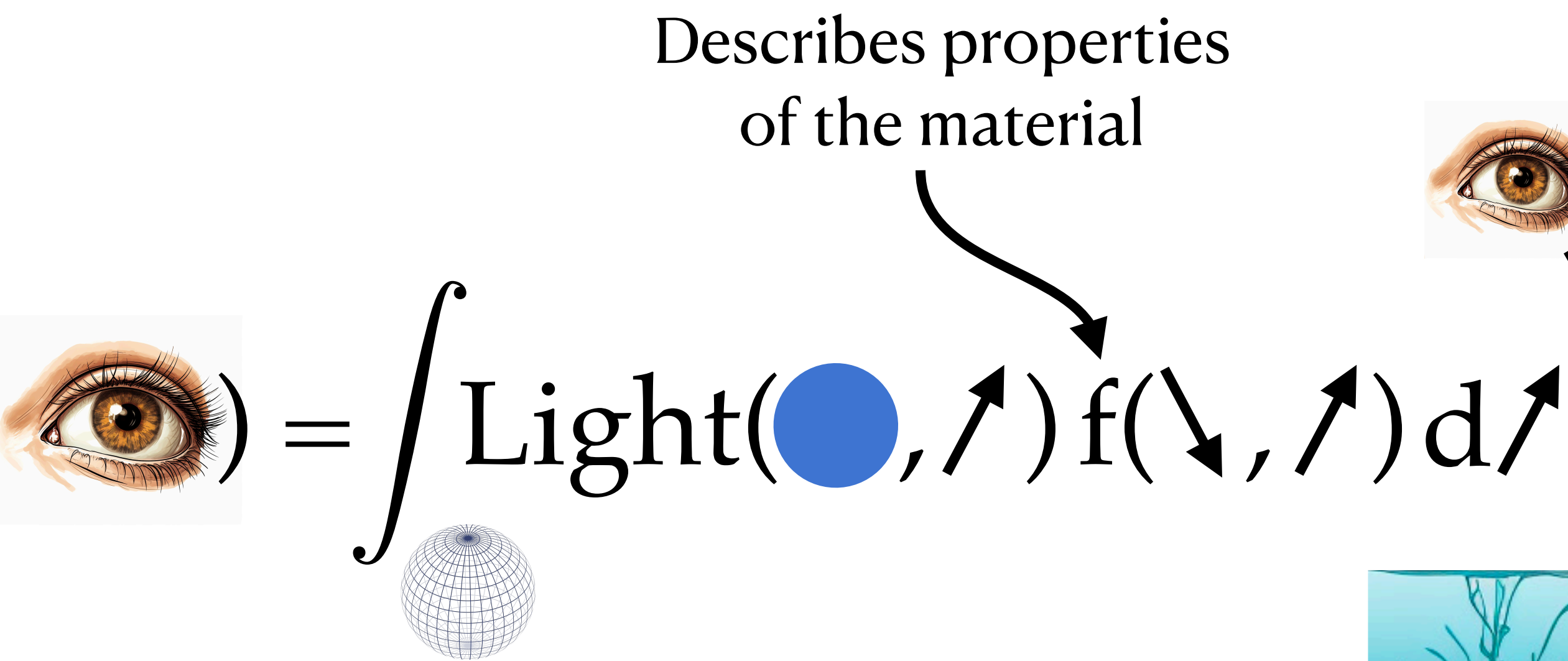


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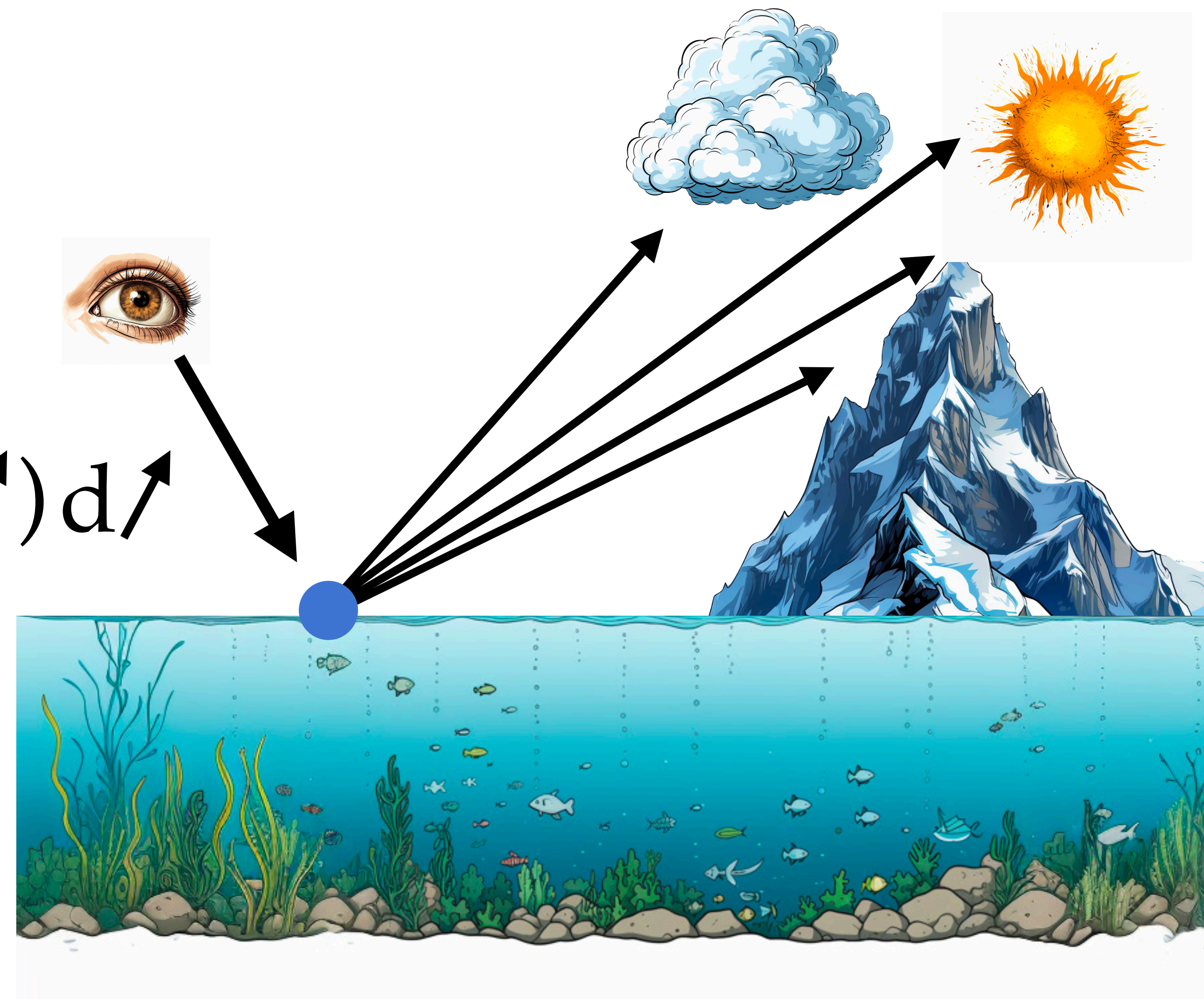
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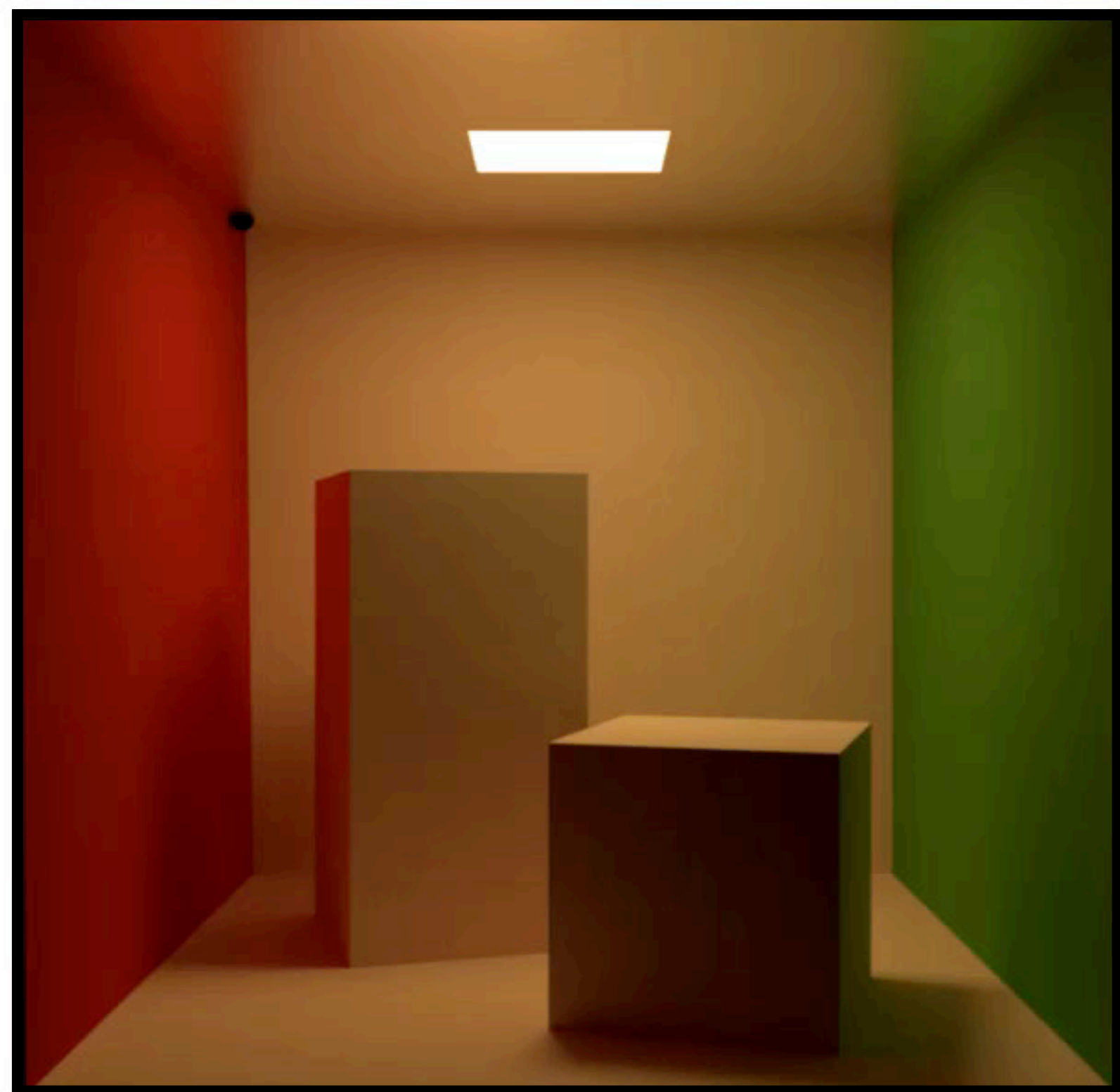
Describes properties of the material



The light reflected by a point is a *weighted average* of the arriving light. This average has infinite resolution and must be computed using an integral.



# Computing integrals to create images

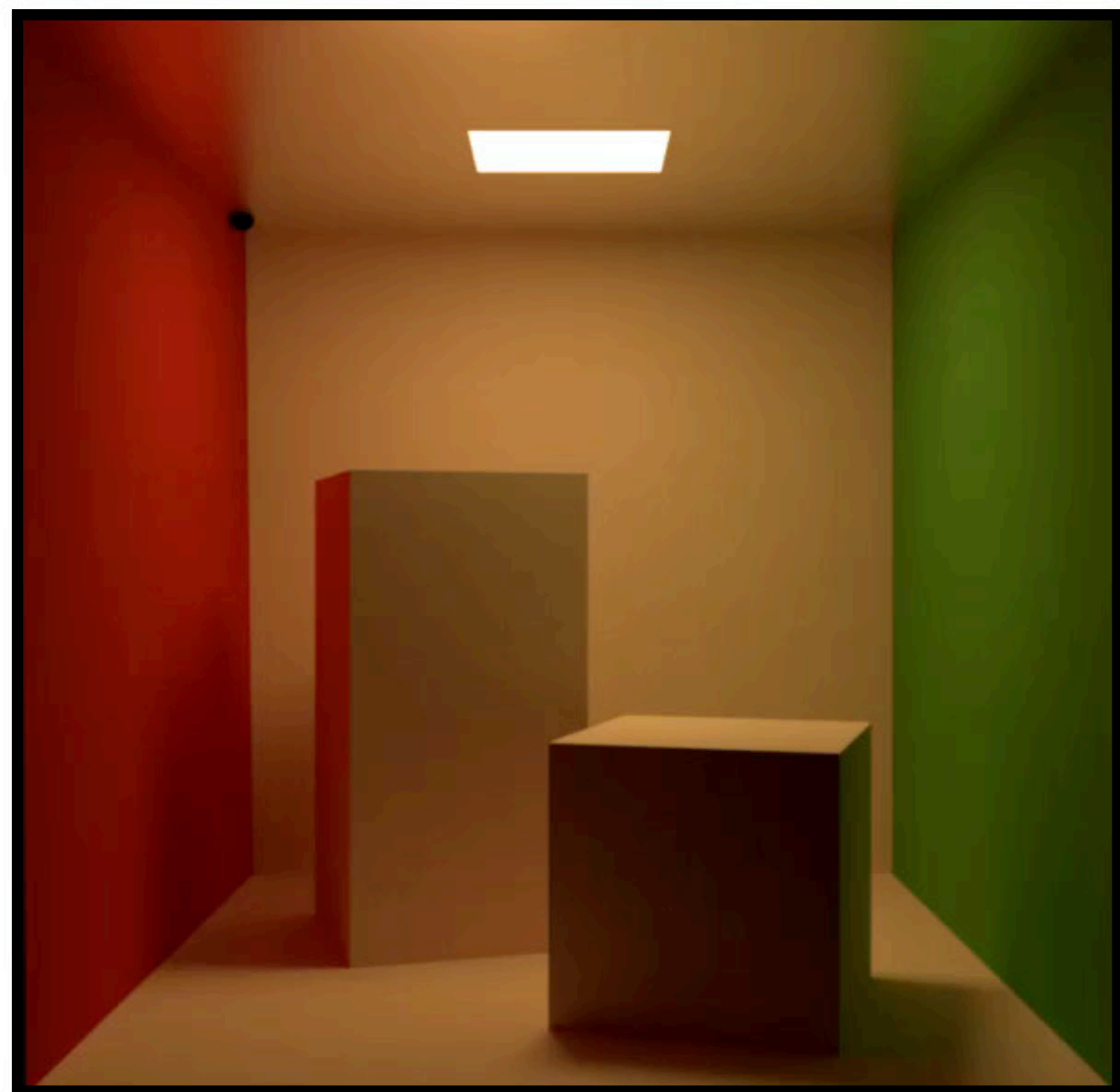


$$\int_{\mathcal{S}^2} \text{[Scene View]} \cdot \text{[Light Source]} d\omega$$

$$= \int_{\mathcal{S}^2} \text{[Final Pixel Color]} d\omega = \text{[Red Square]}$$

Final pixel color

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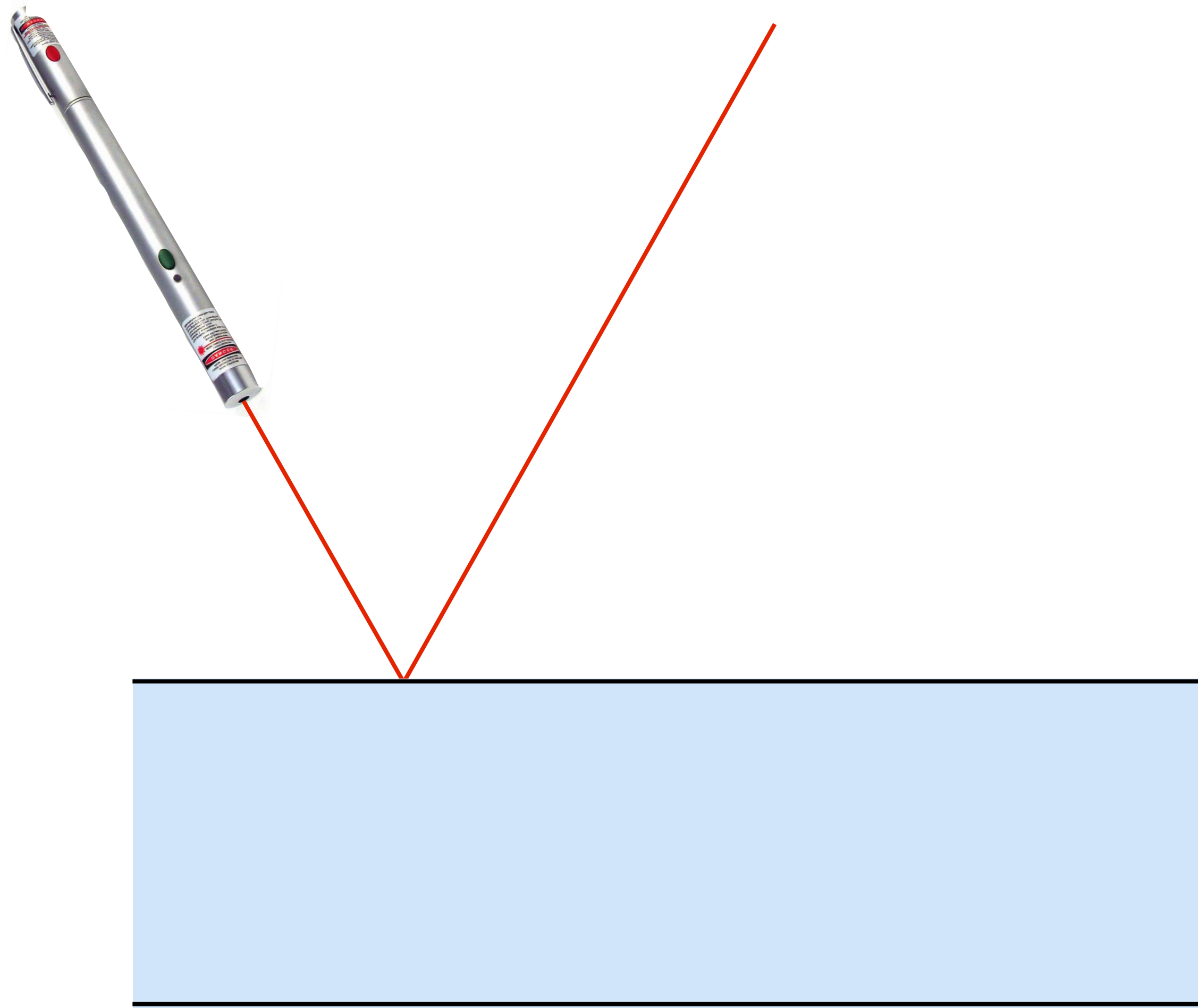
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# Various types of materials and the effects they produce



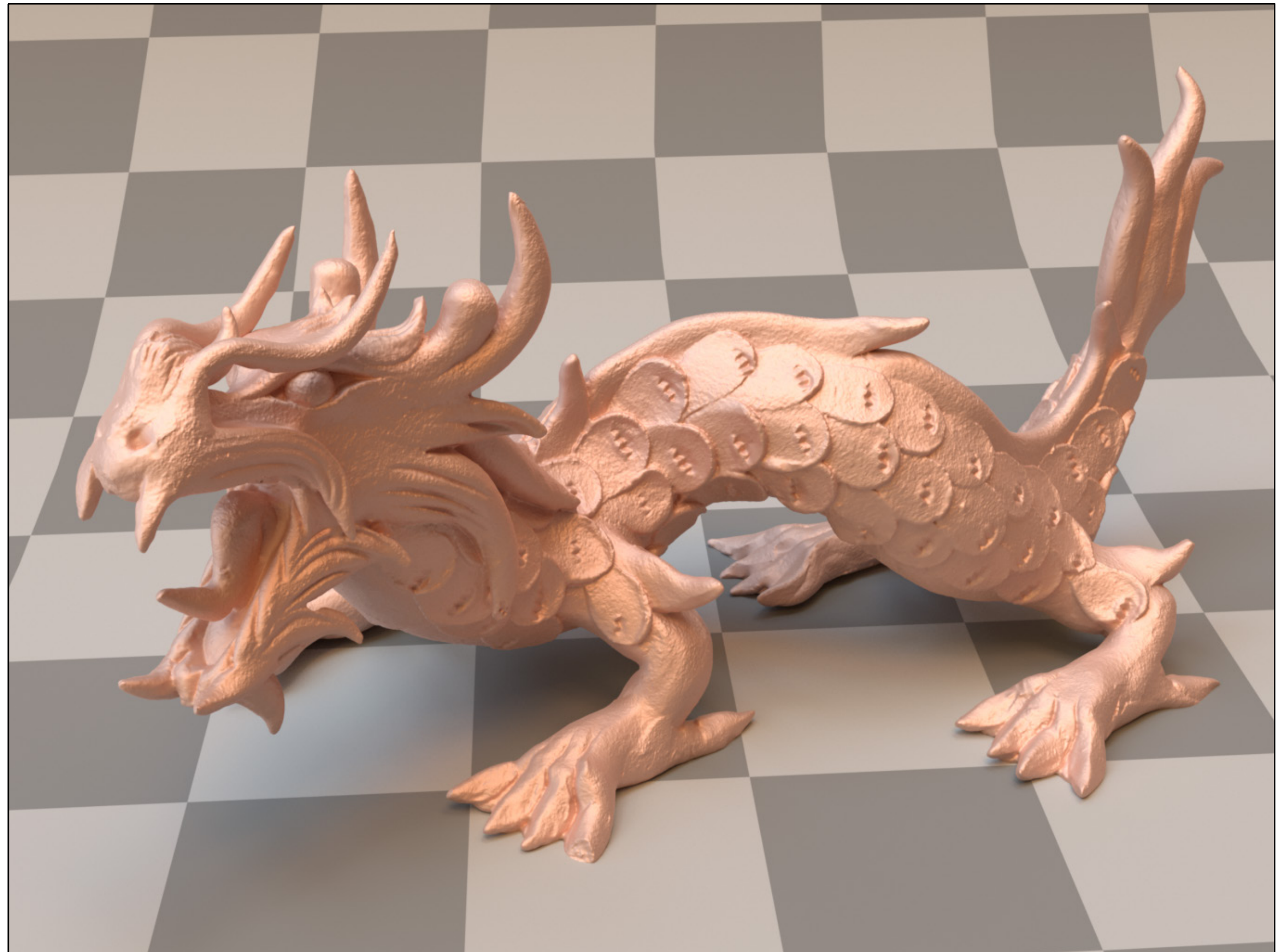
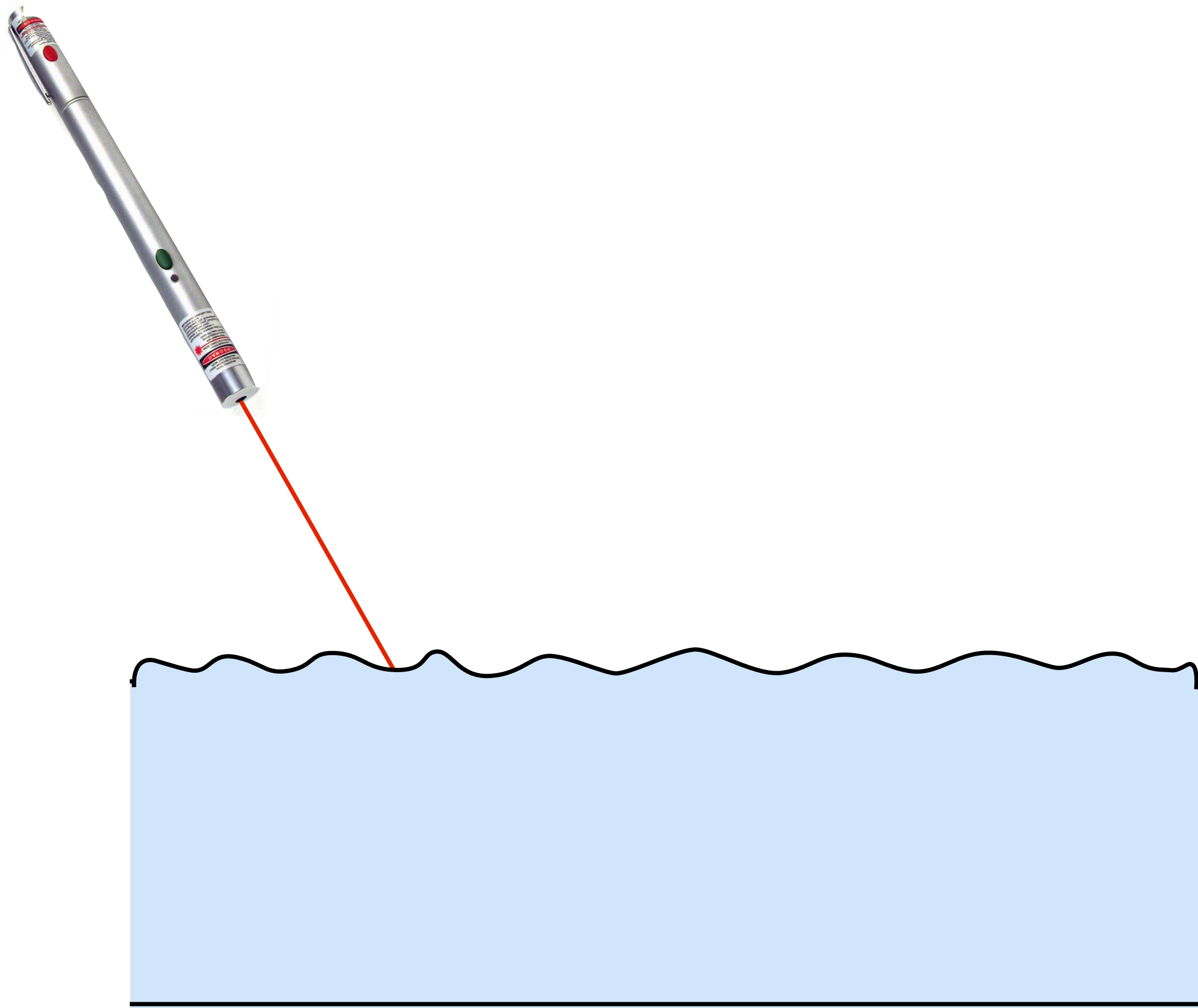
[Rendered image made using Mitsuba Renderer]

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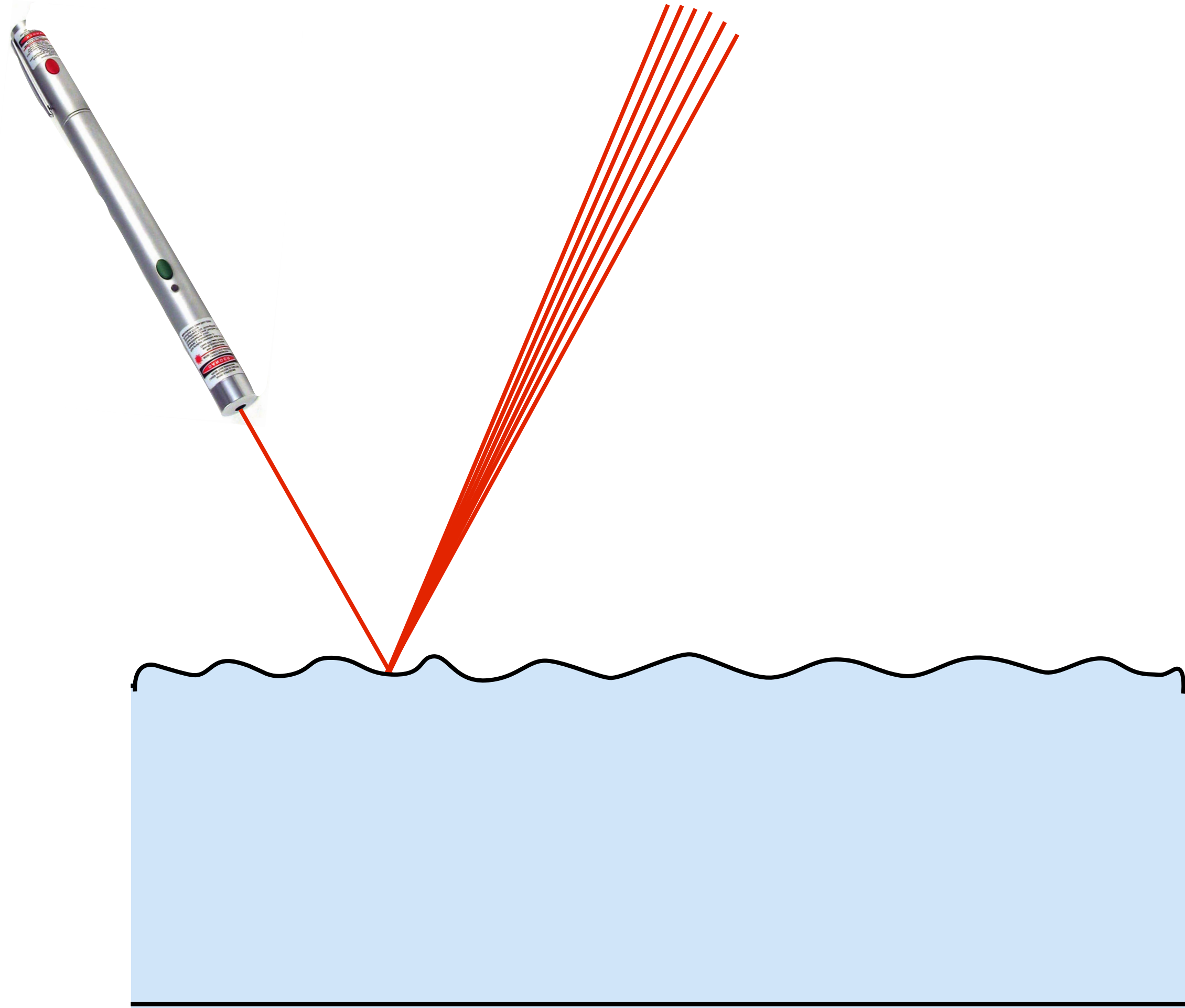
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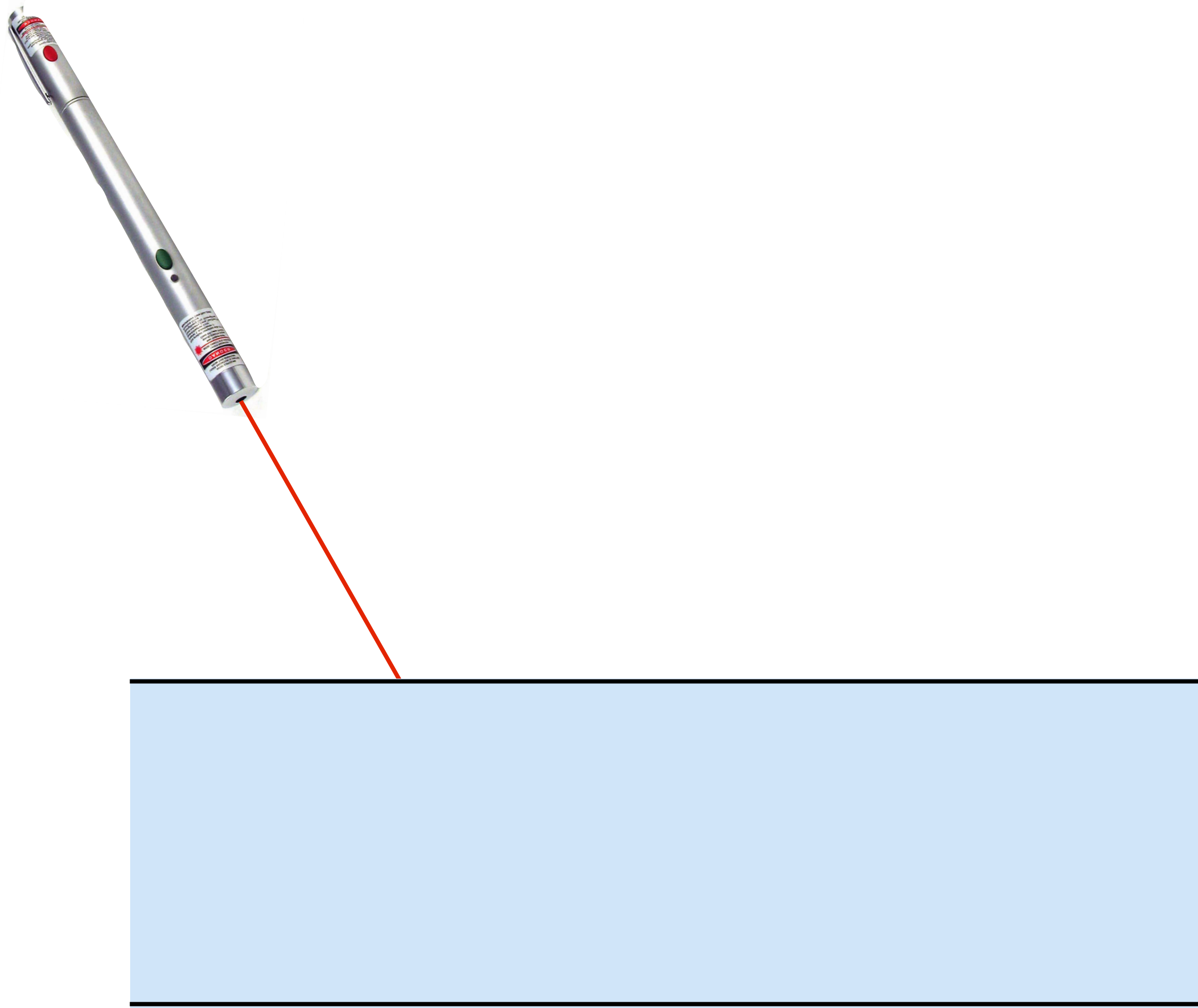
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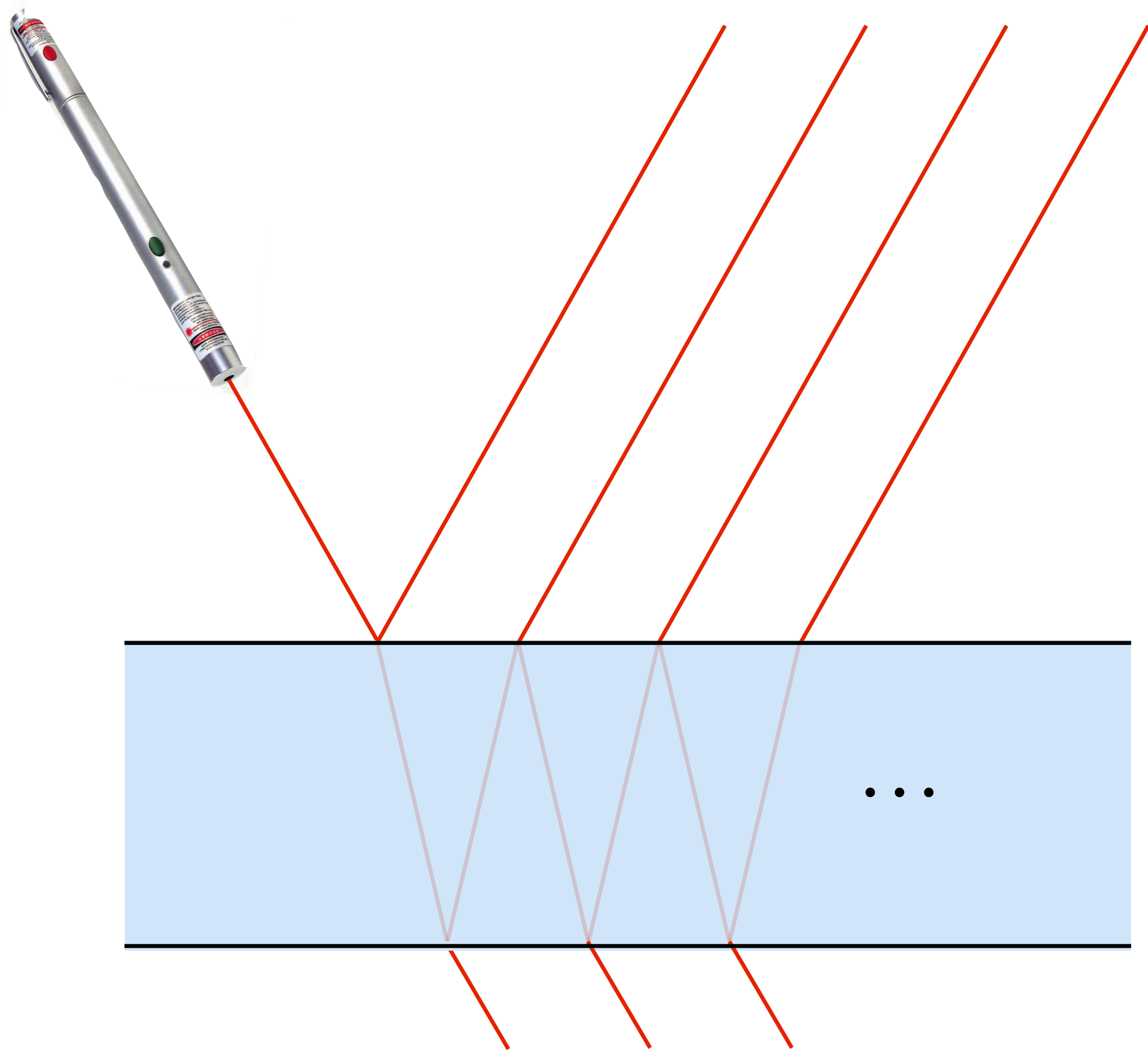
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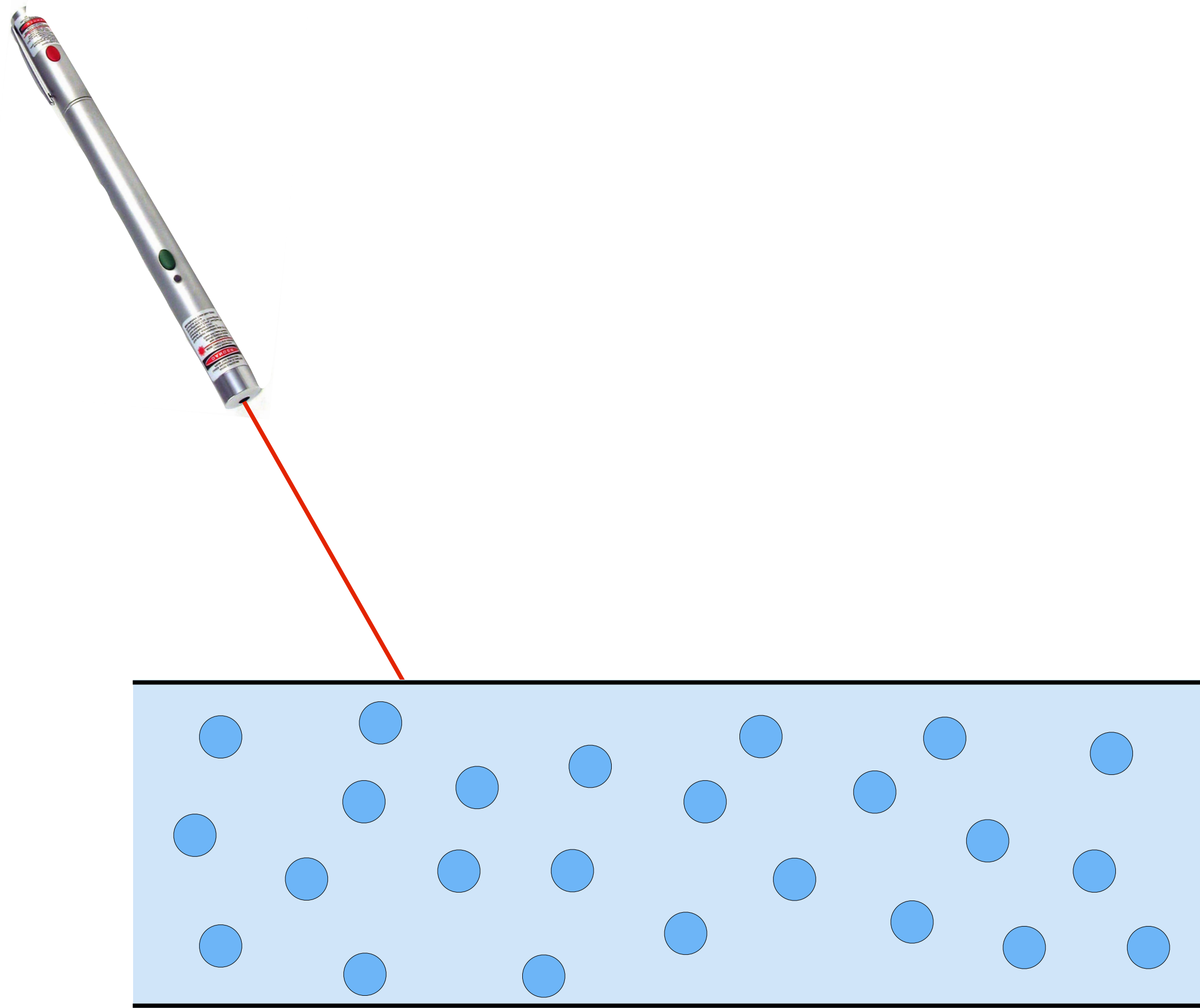
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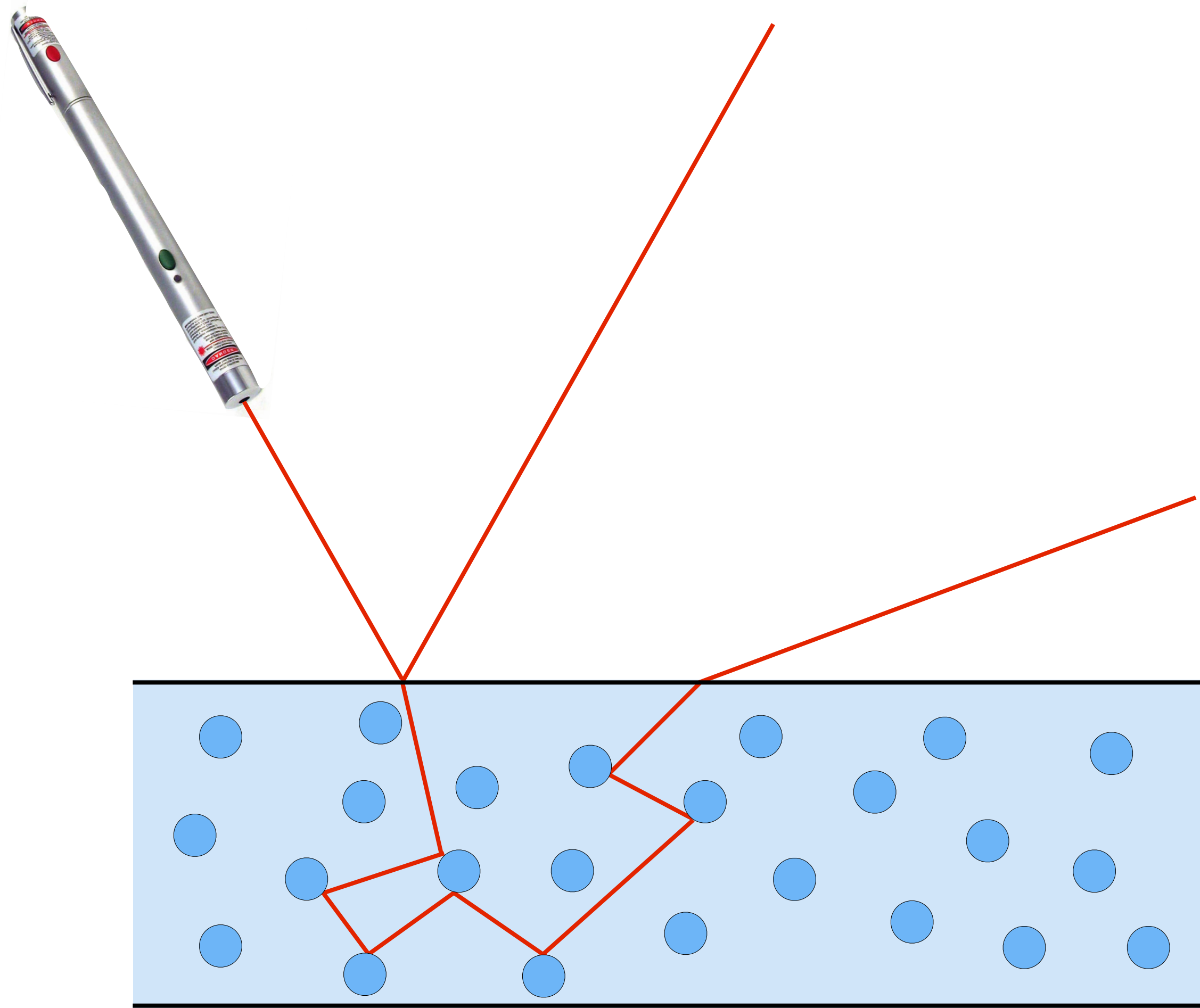
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Dawn of the Planet of the Apes , © 2013 Twentieth Century Fox Film Corporation. All Rights Reserved.  
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  - Riemann integration
  - Rules: midpoint, trapezoid, Simpson.
  - The curse of dimensionality

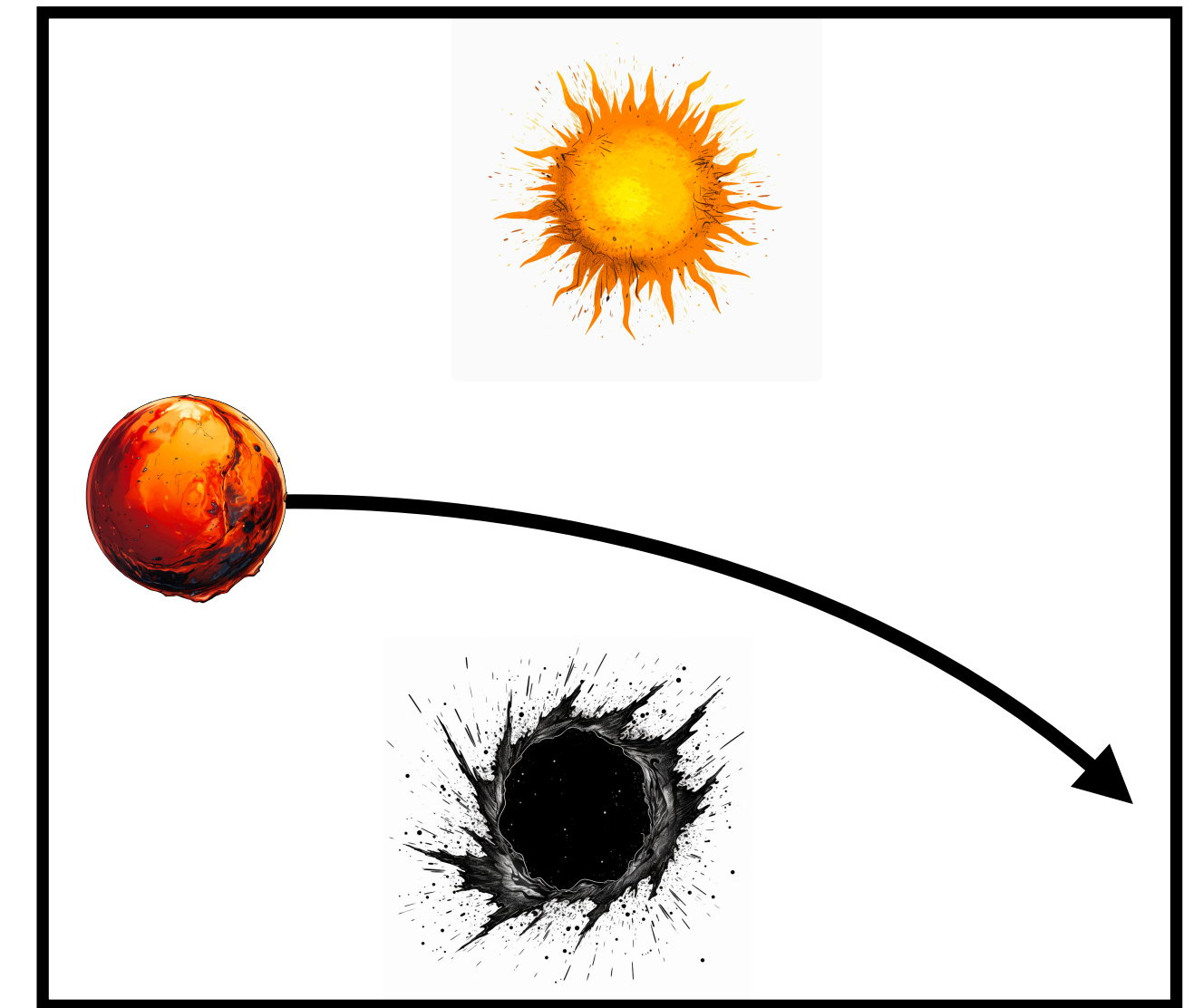
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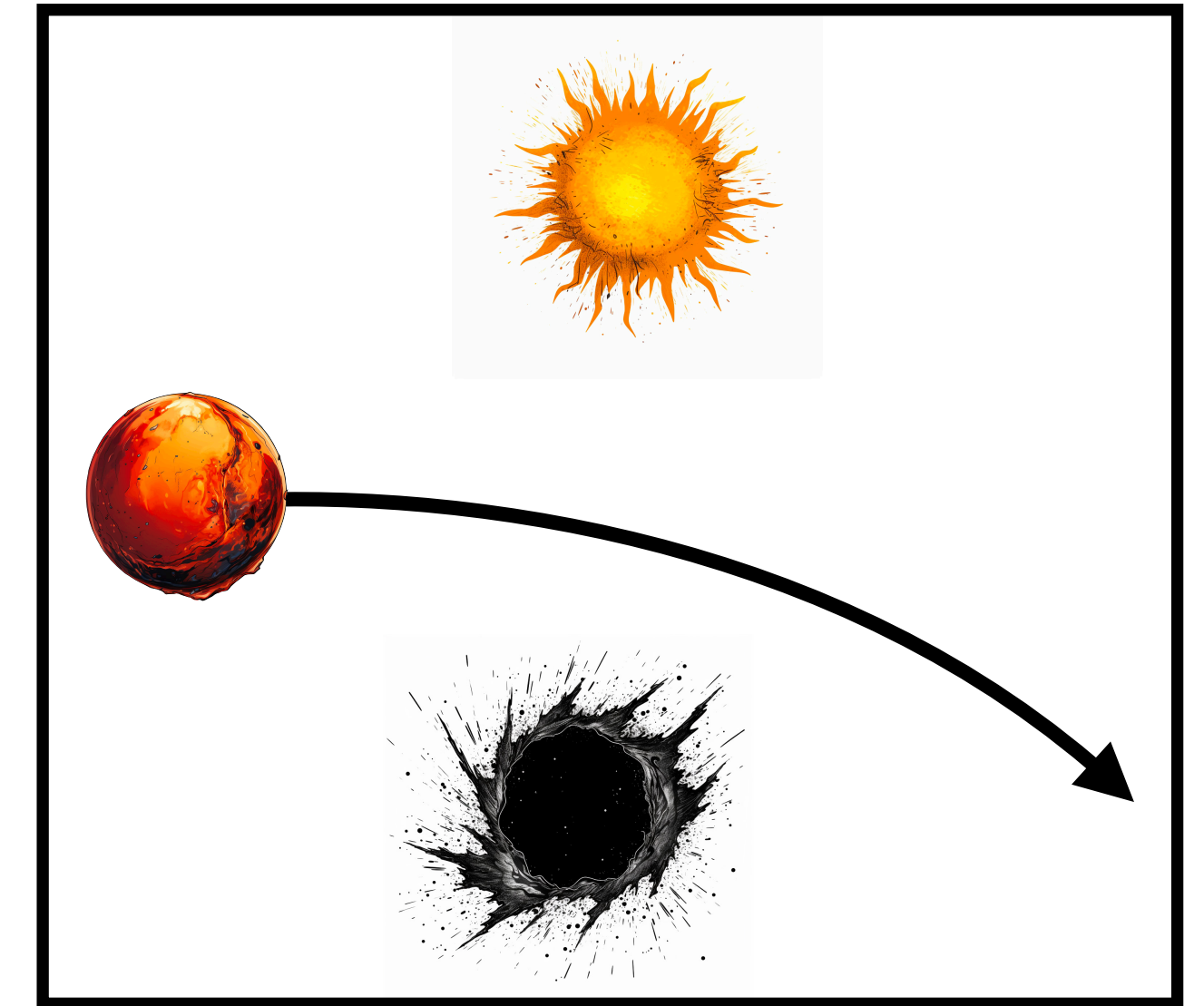
*Low  
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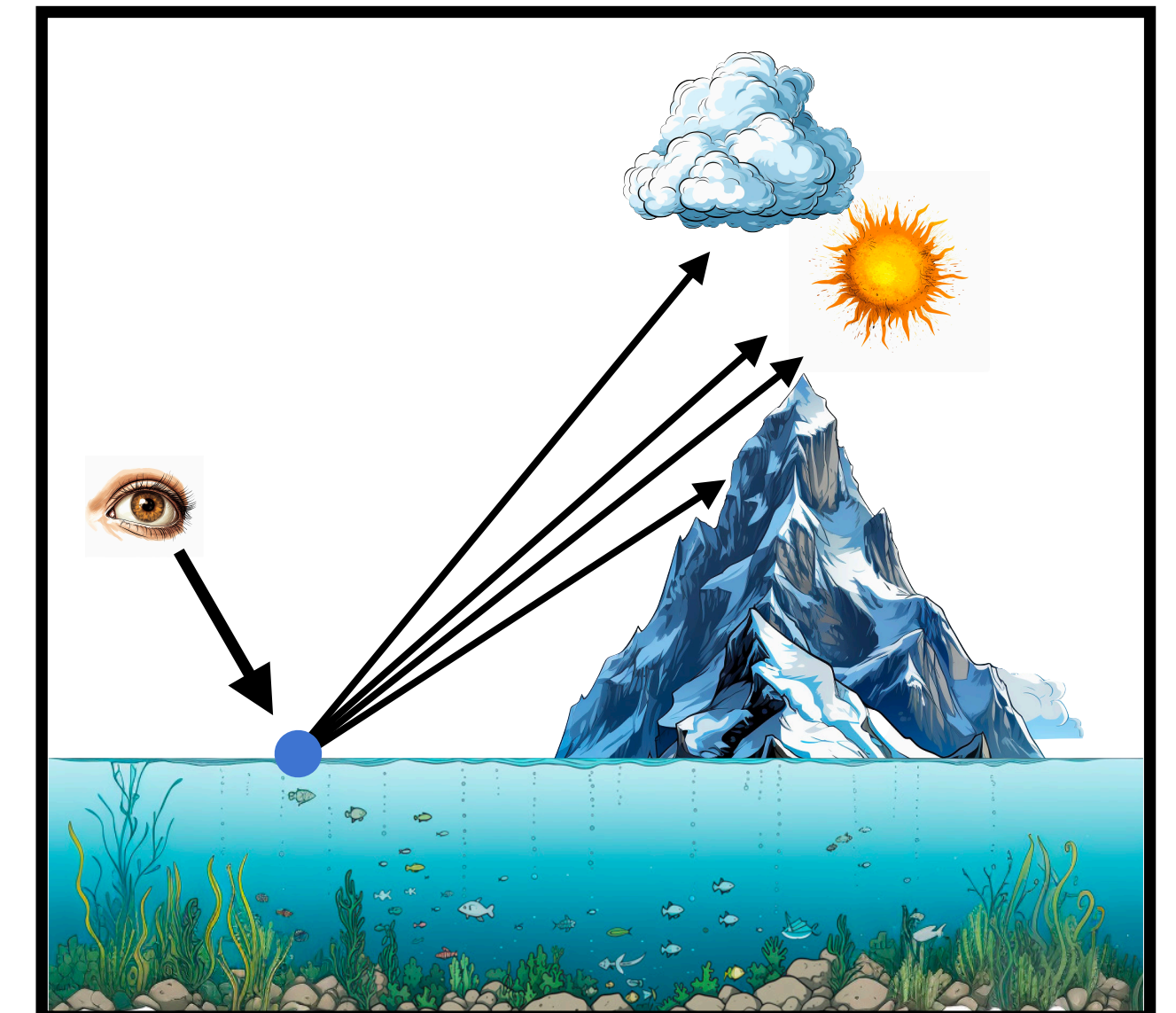
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$$F(x) := \int f(x) dx$$

*(indefinite integral)*

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2. Apply fundamental theorem of calculus

$$\int_a^b f(x) dx = F(b) - F(a)$$

*(definite integral)*

# The rules of the game, aka. calculus

**Rules for specific functions:**

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## Rules for specific functions:

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C \quad (\text{for } a \neq -1)$$

$$\int \frac{1}{x} dx = \ln |x| + C \quad \int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$\int \sin x dx = -\cos x + C$$

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$$\int \sin x dx = -\cos x + C$$

## Meta-rules: integration by parts:

$$\begin{aligned} \int_a^b u(x)v'(x) dx &= [u(x)v(x)]_a^b - \int_a^b u'(x)v(x) dx \\ &= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x) dx \end{aligned}$$

# Integration versus differentiation

$$\int f(x) dx$$

$$f(x)$$

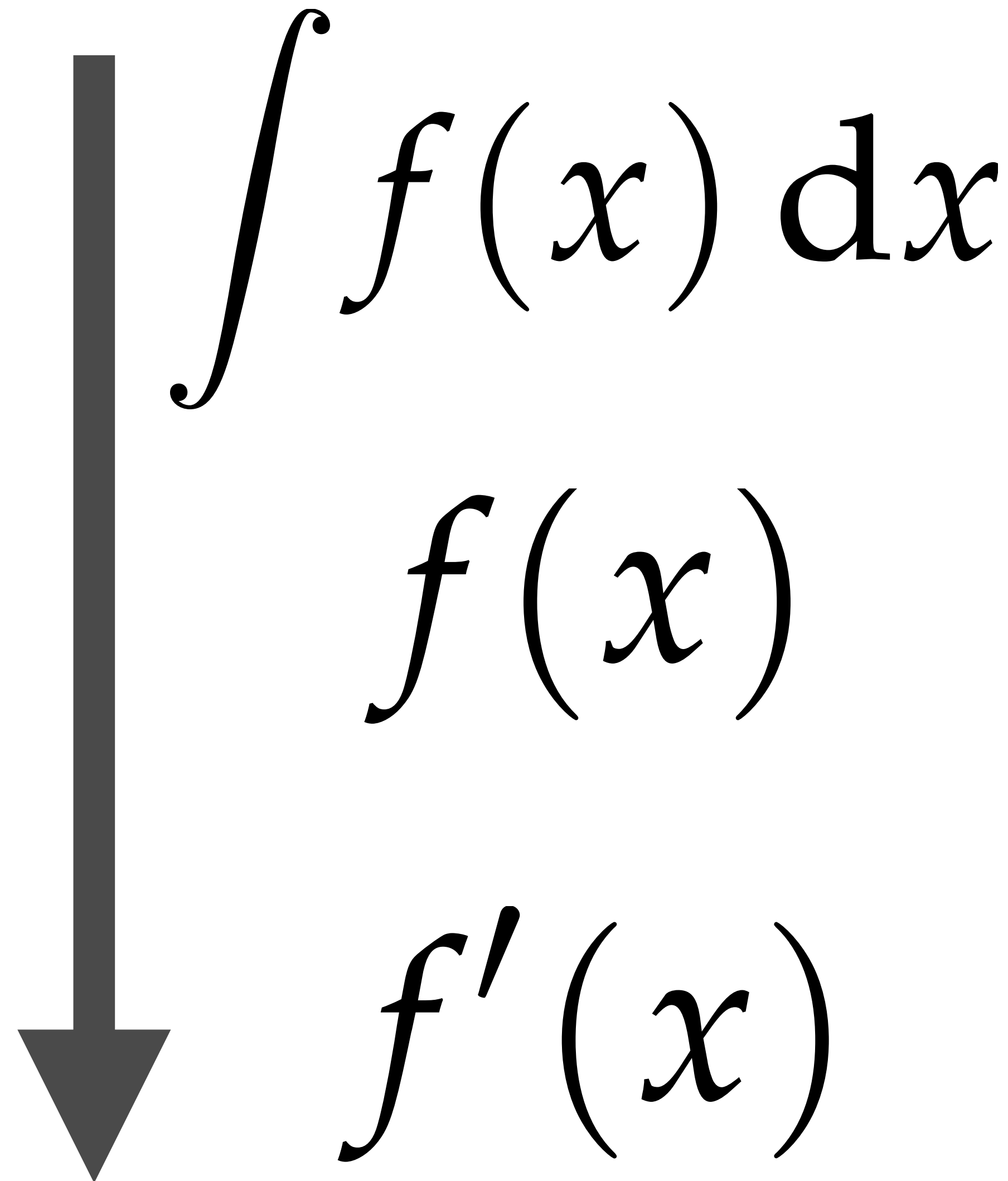
$$f'(x)$$

# Integration versus differentiation

## Differentiation

Analytic &  
numerical  
approaches

**OK**



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$$\int f(x) dx$$

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Analytic  
approaches  
**almost  
always fail**

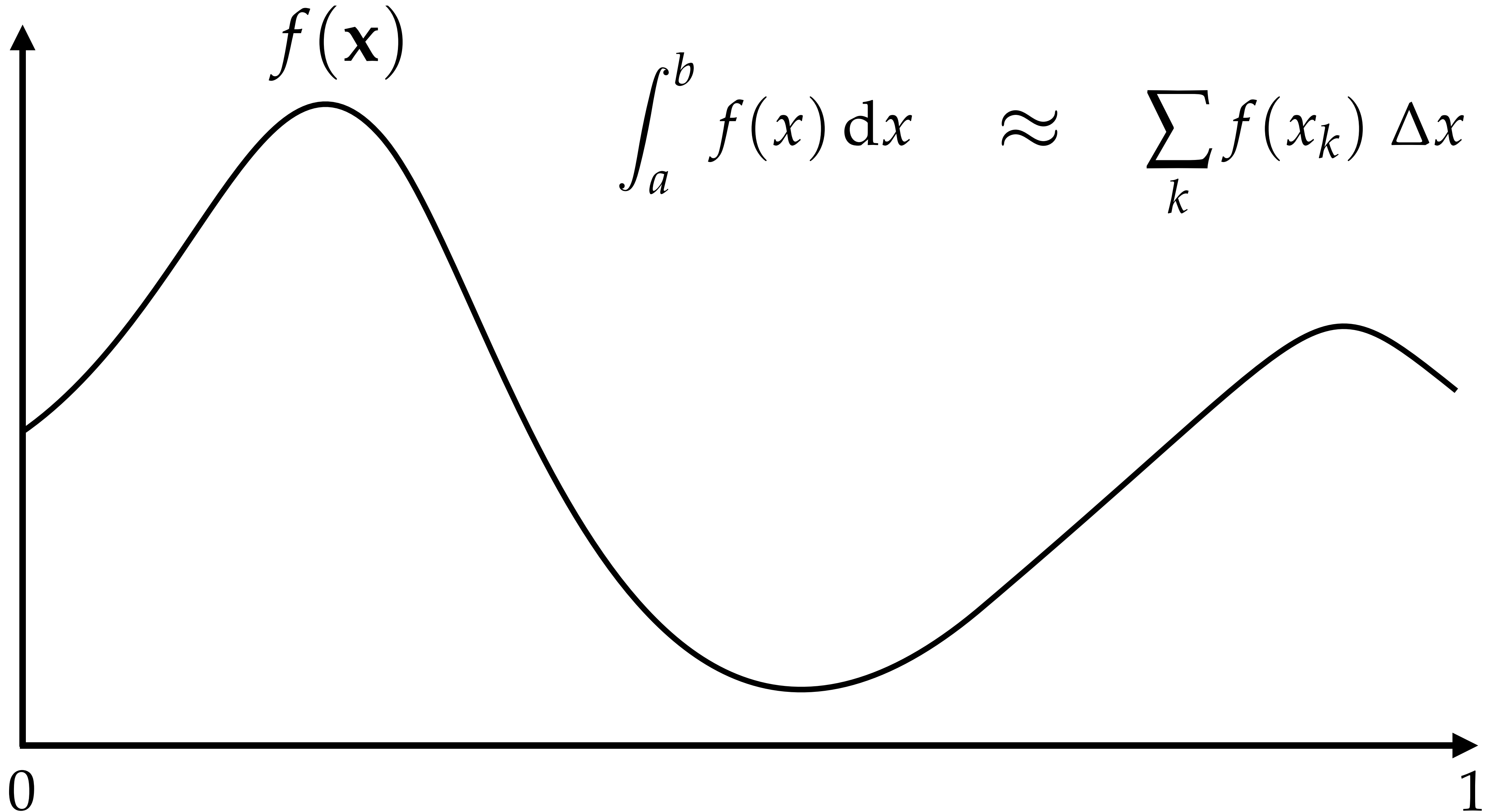


Numerical  
approaches

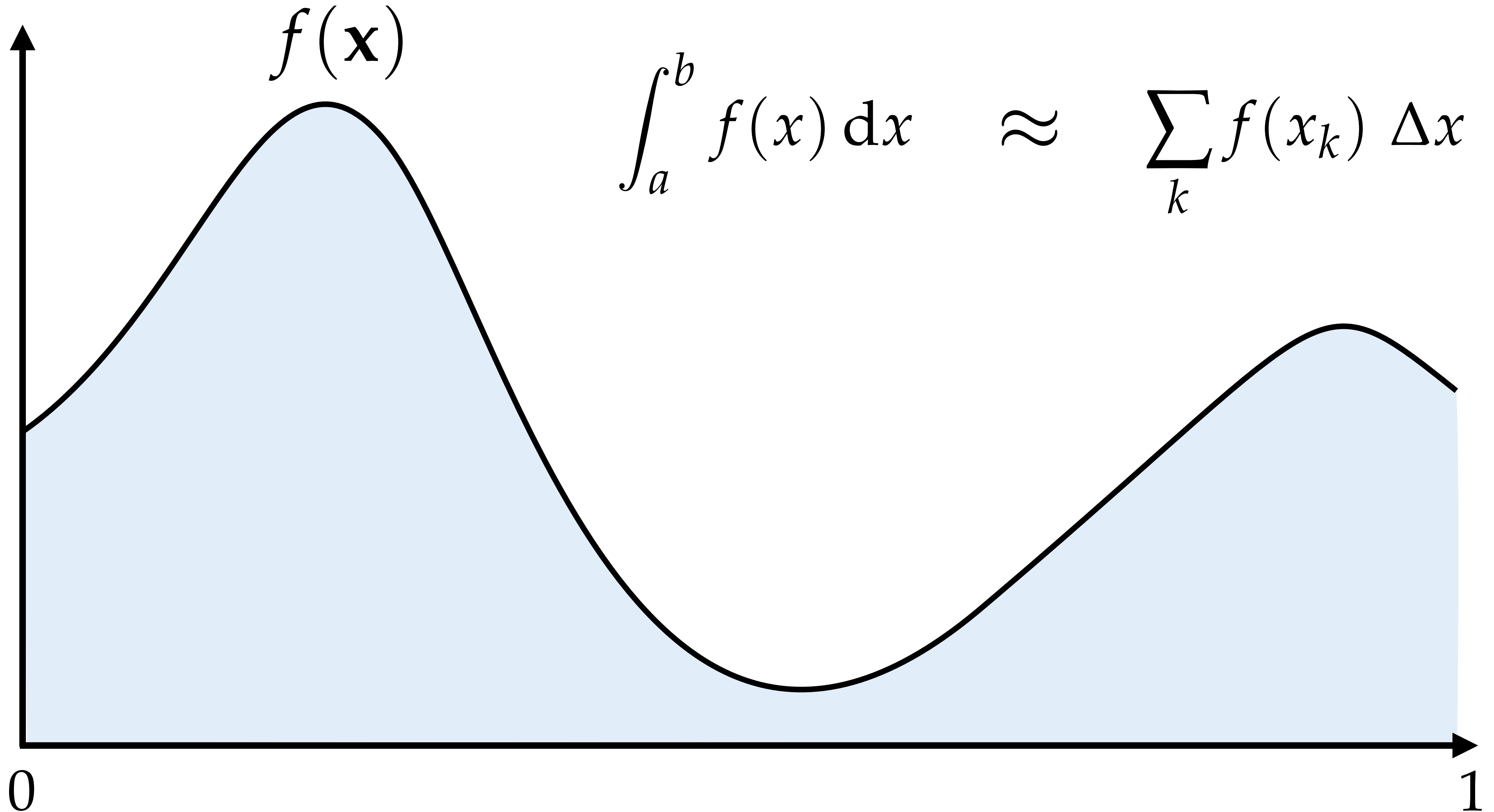
**OK**



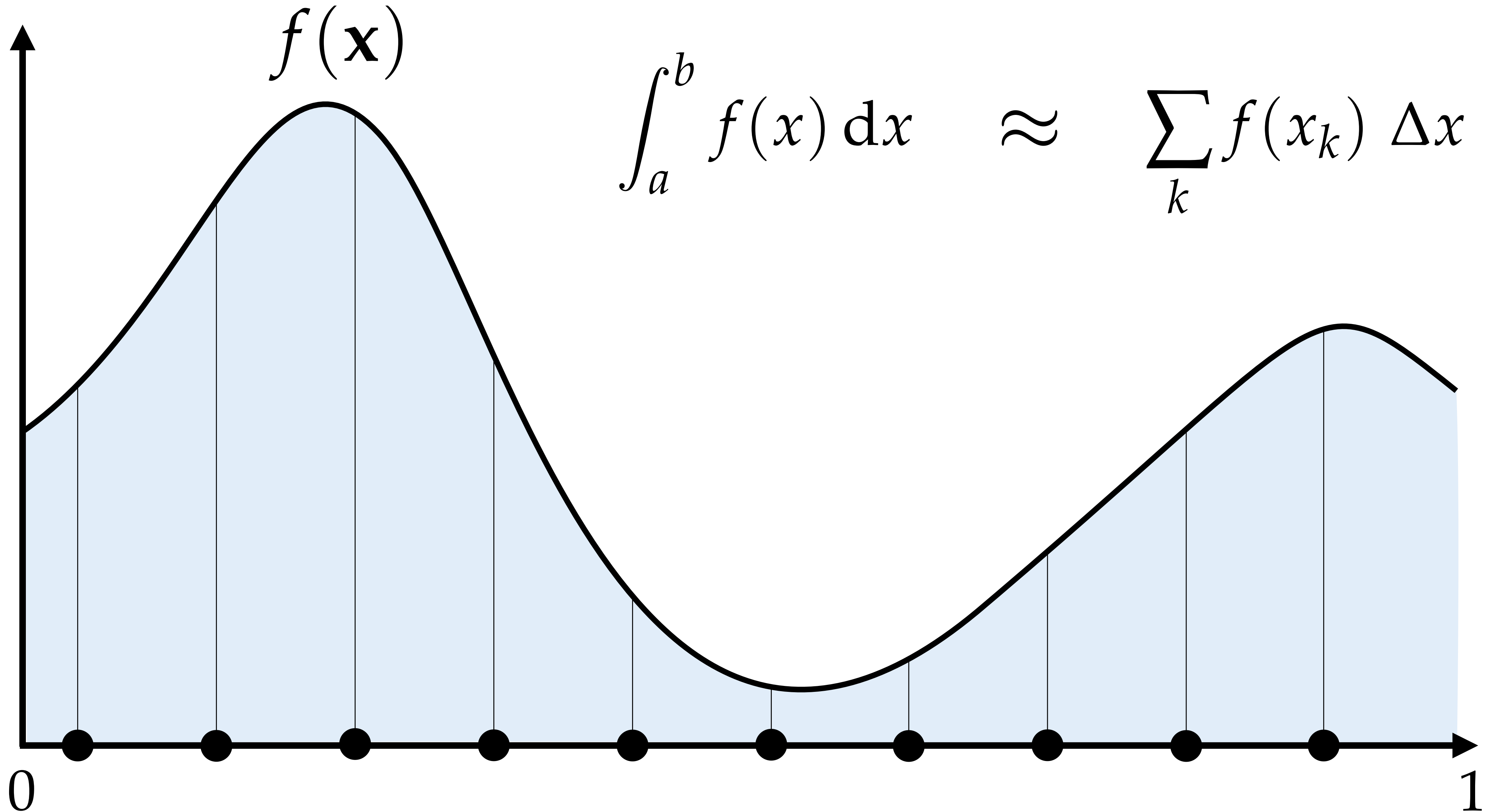
# Approximating integrals using Riemann sums



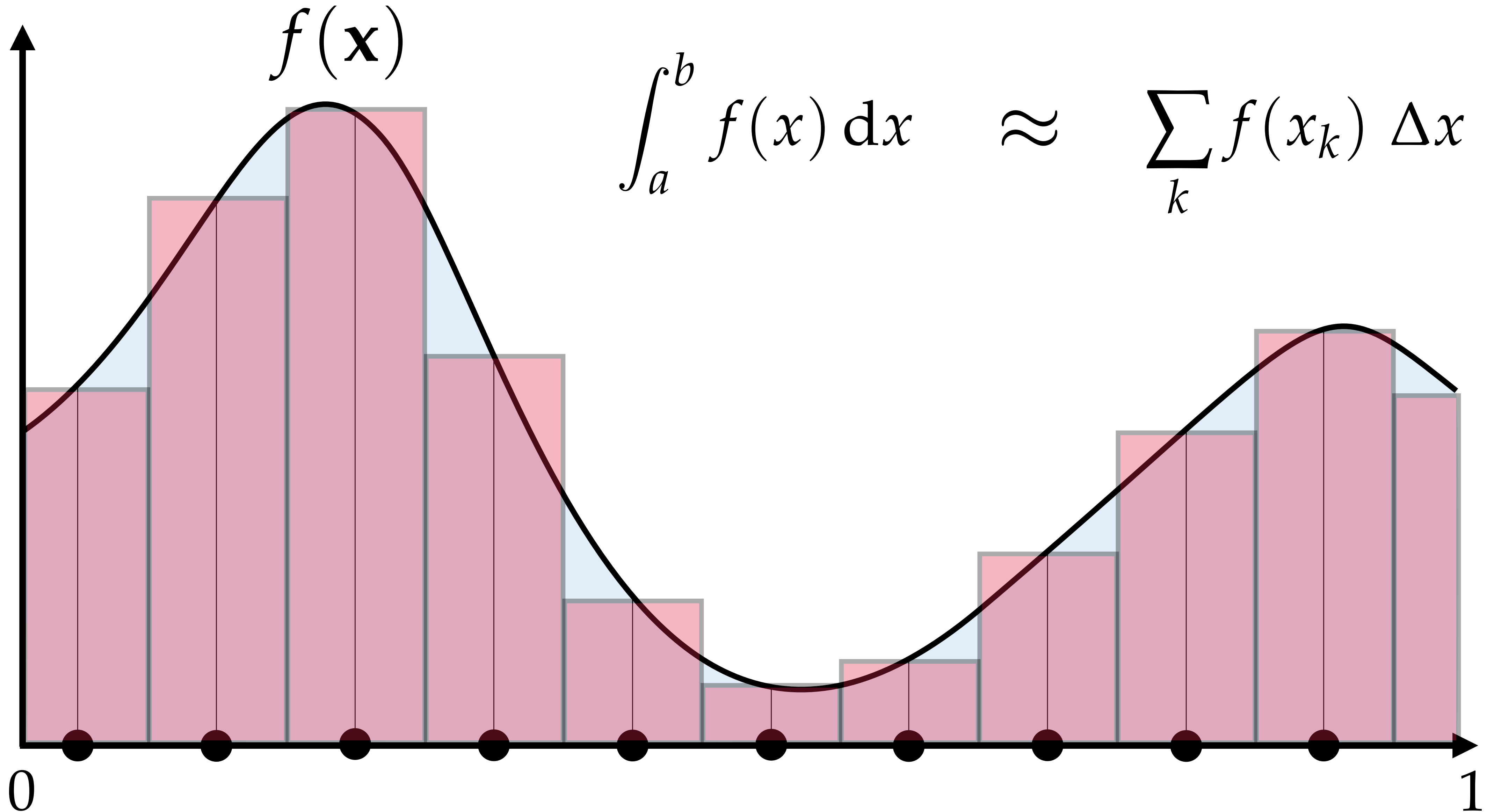
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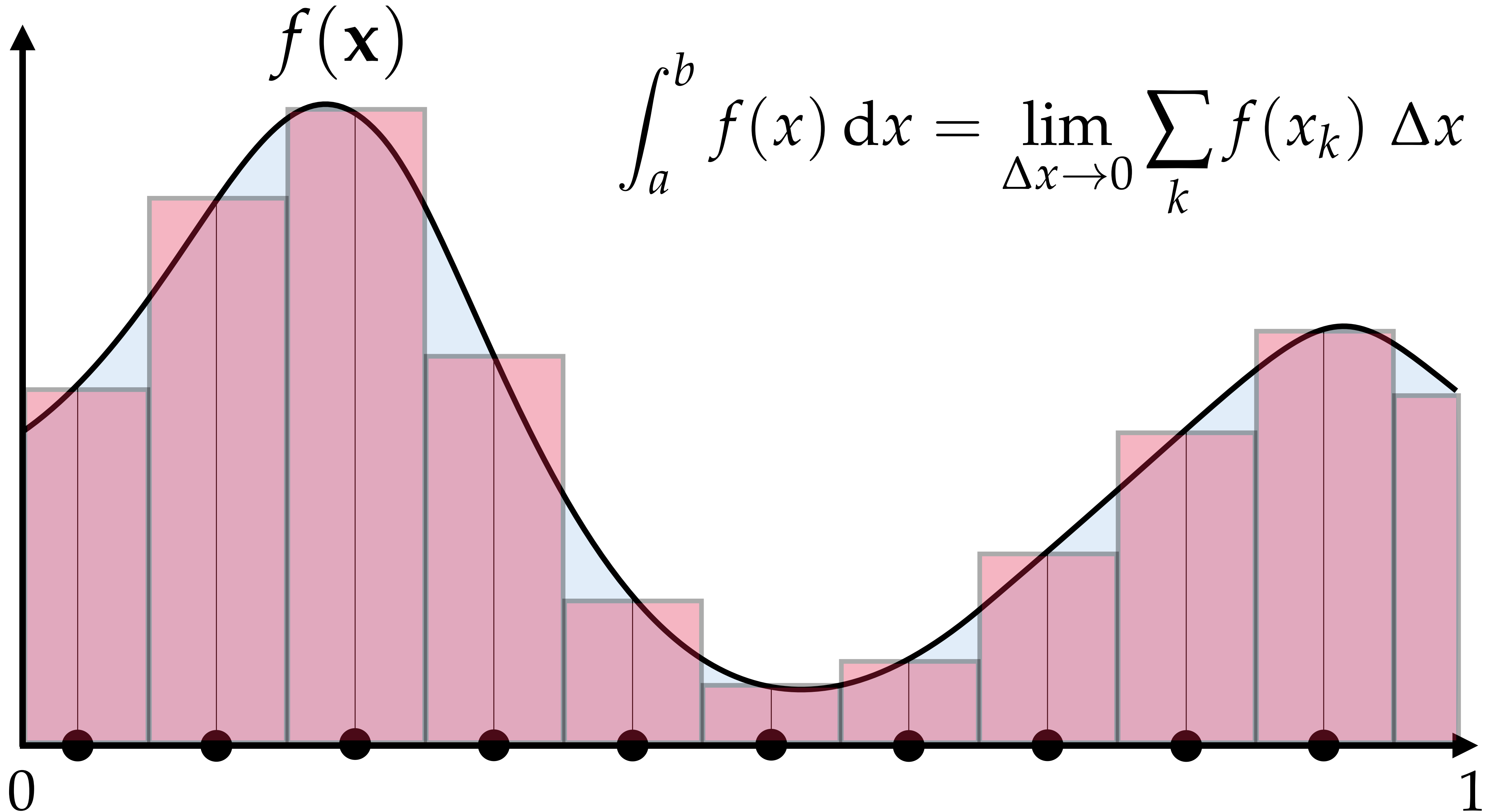
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# Quadrature

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- **Two flavors:**
  - **Quadrature rule:** applies to small part of the function
  - **Composite quadrature:** repeatedly applies the rule to integrate the *entire* function.

# Midpoint rule

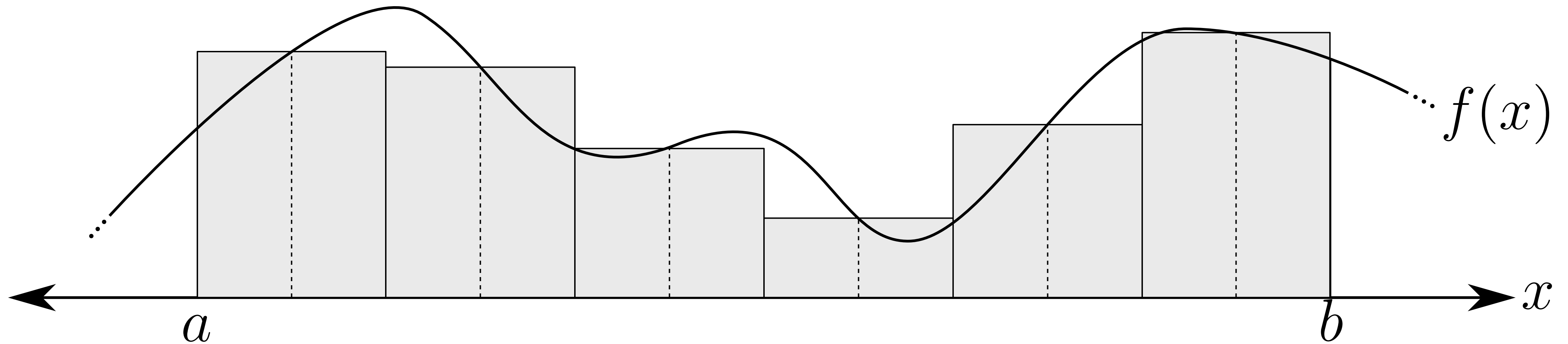
$$\int_a^b f(x) dx \approx (b - a) f\left(\frac{a + b}{2}\right)$$

$$f\left(\frac{a + b}{2}\right)$$



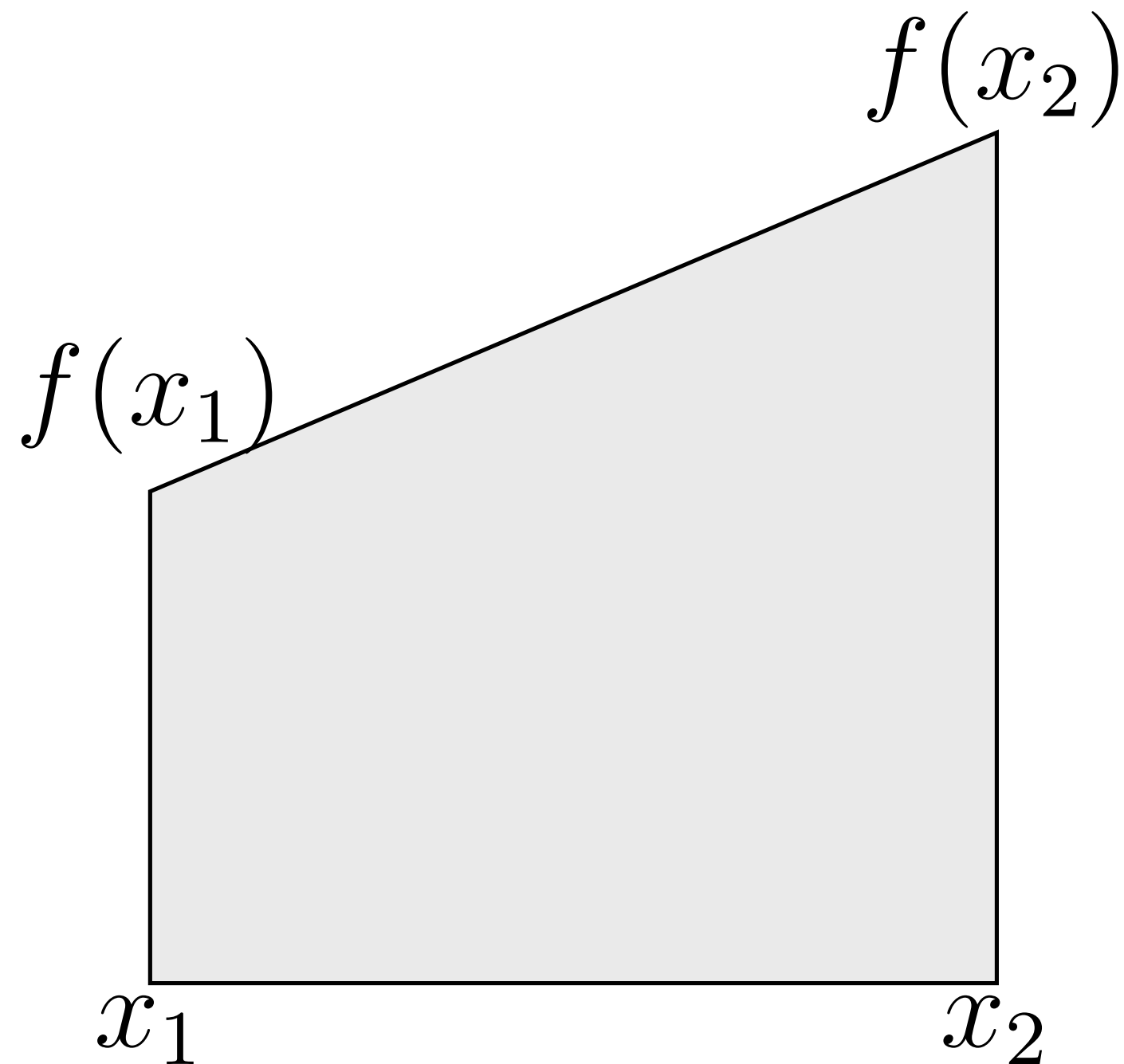
# Composite midpoint rule

$$\int_a^b f(x) dx \approx \sum_{i=1}^k f\left(\frac{x_{i+1} + x_i}{2}\right) \Delta x$$



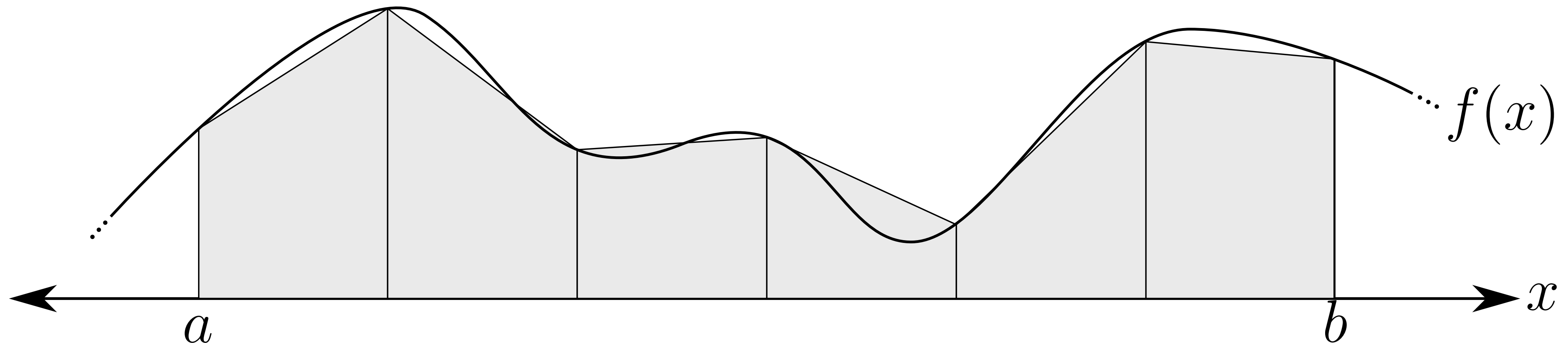
# Trapezoid rule

$$\int_a^b f(x) dx \approx (b - a) \frac{f(a) + f(b)}{2}$$



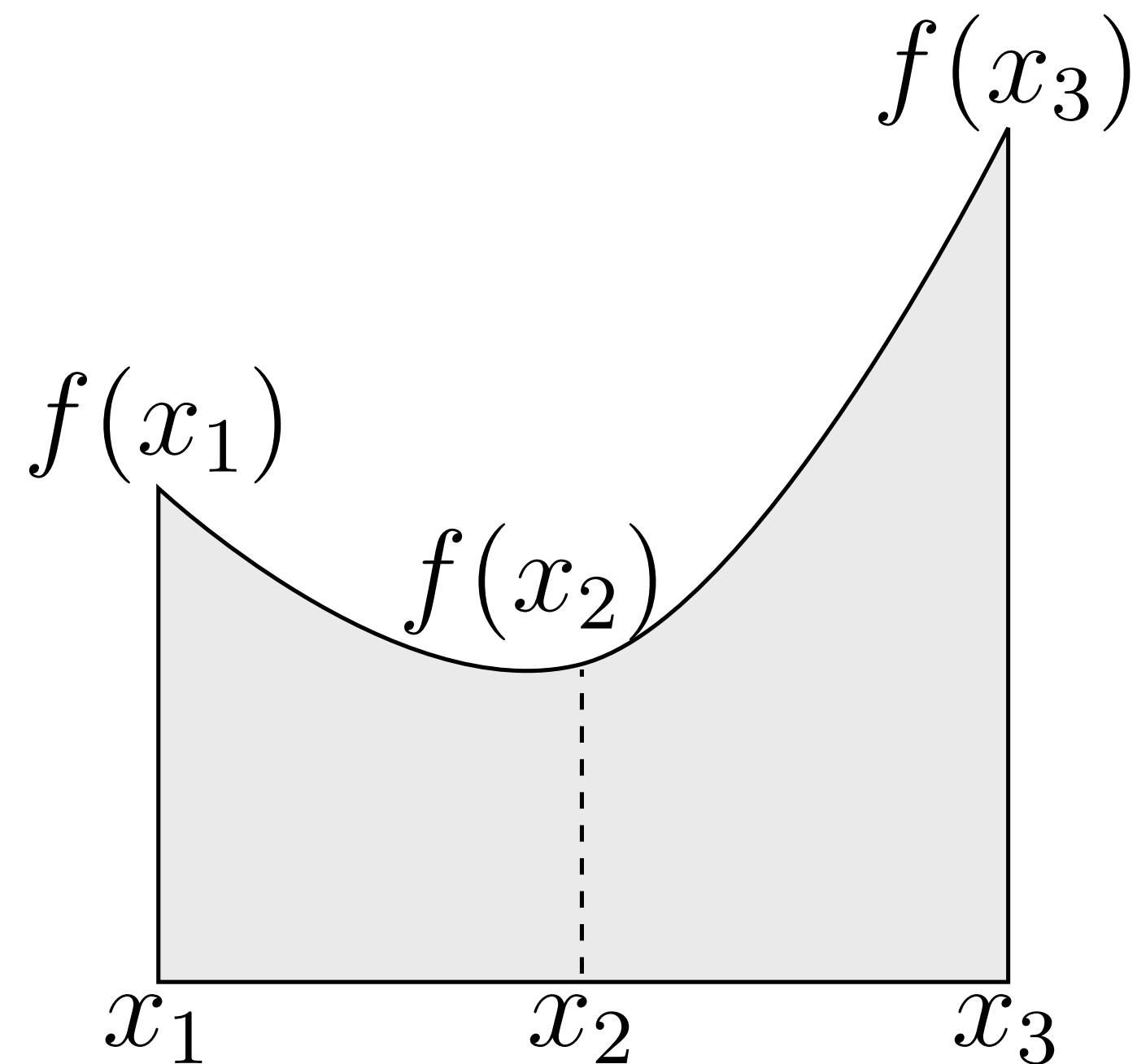
# Composite trapezoid rule

$$\int_a^b f(x) dx \approx \sum_{i=1}^k \left( \frac{f(x_i) + f(x_{i+1})}{2} \right) \Delta x$$
$$= \Delta x \left( \frac{1}{2} f(a) + f(x_1) + \dots + f(x_{k-1}) + \frac{1}{2} f(b) \right)$$



# Simpson's rule

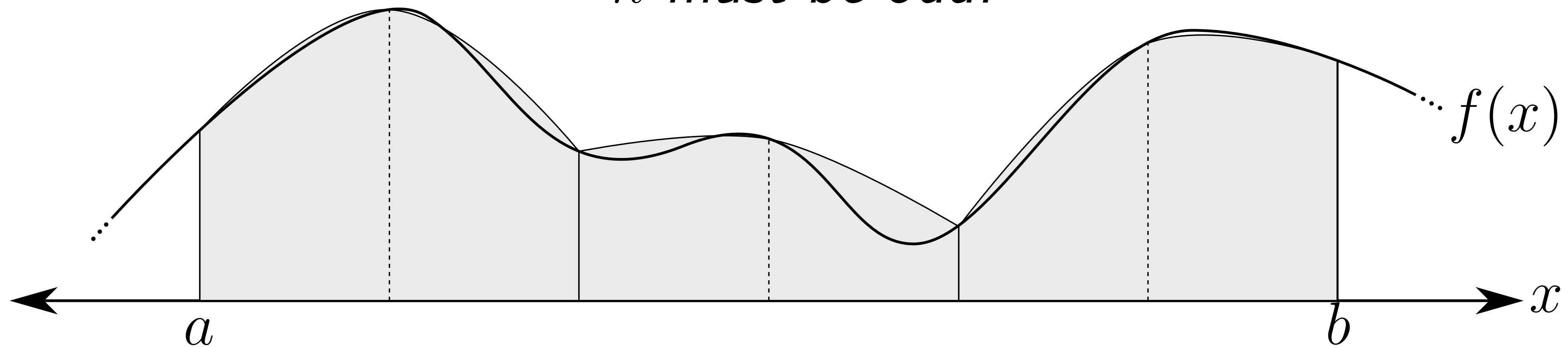
$$\int_a^b f(x) dx \approx \frac{b-a}{6} \left( f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)$$



# Composite Simpson's rule

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3} \left[ f(a) + 2 \sum_{i=1}^{n-2-1} f(x_{2i}) + 4 \sum_{i=1}^{n/2} f(x_{2i-1}) + f(b) \right]$$
$$= \frac{\Delta x}{3} [f(a) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(b)]$$

*n must be odd!*



**Demo time**

# Adaptive quadrature

*# High-level idea*

```
def integrate(f, a, b):
```

```
    i_t = trapezoid(f, a, b)
```

```
    i_s = simpson(f, a, b)
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```
    if abs(i_s - i_t) >  $\epsilon$  * abs(i_t):
```

```
        mid = (a + b) / 2
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```
        return integrate(f, a, mid) + integrate(f, mid, b)
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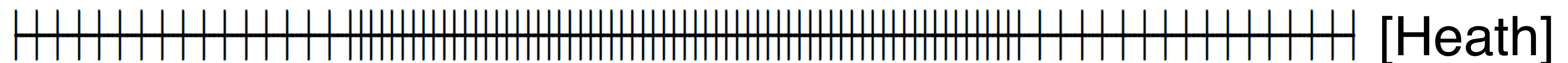
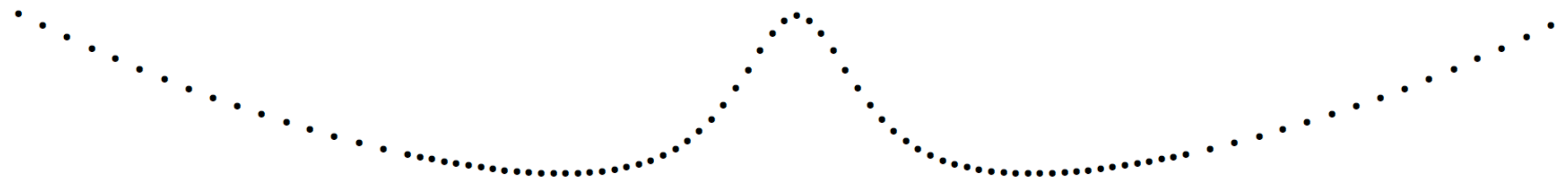
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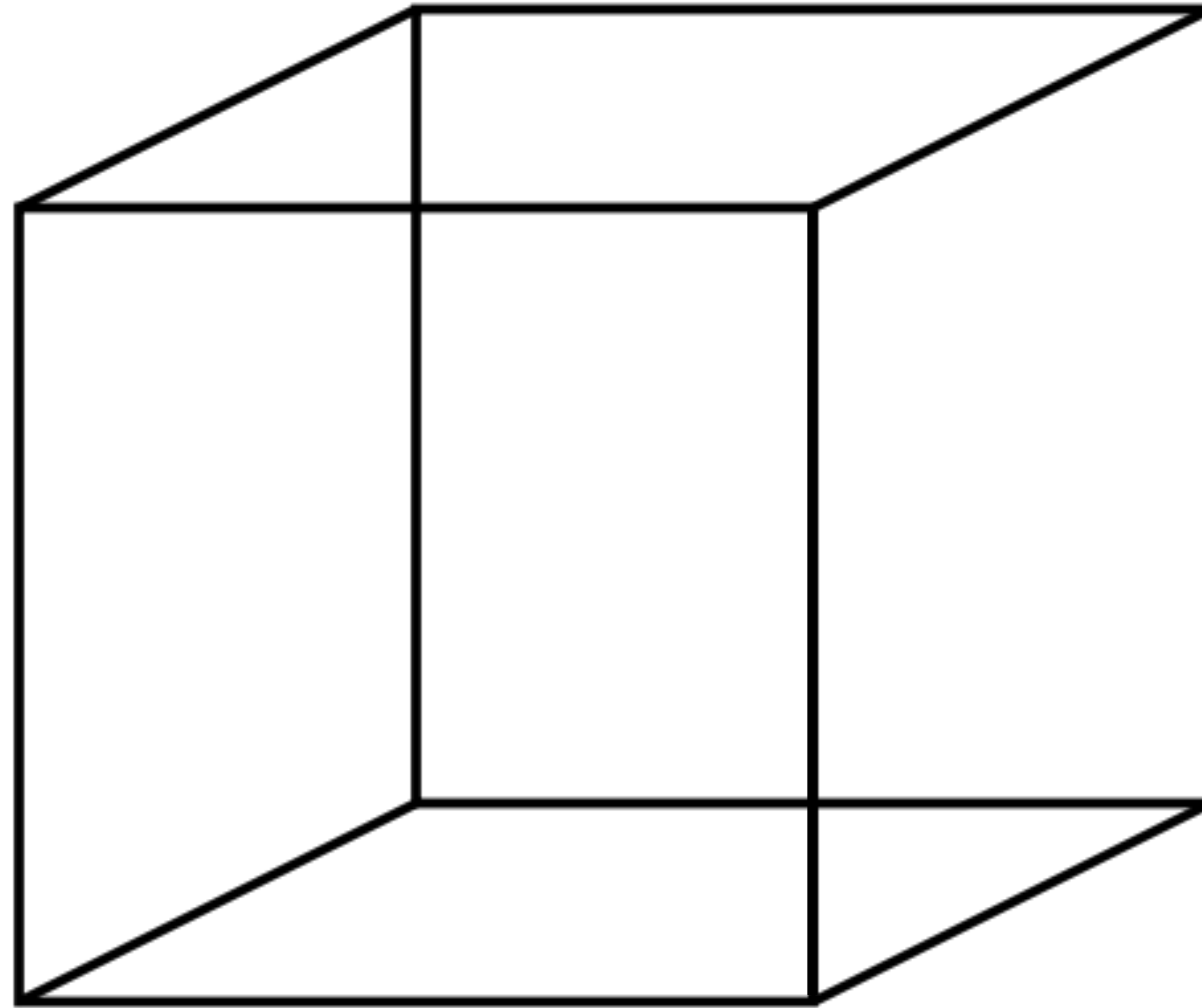
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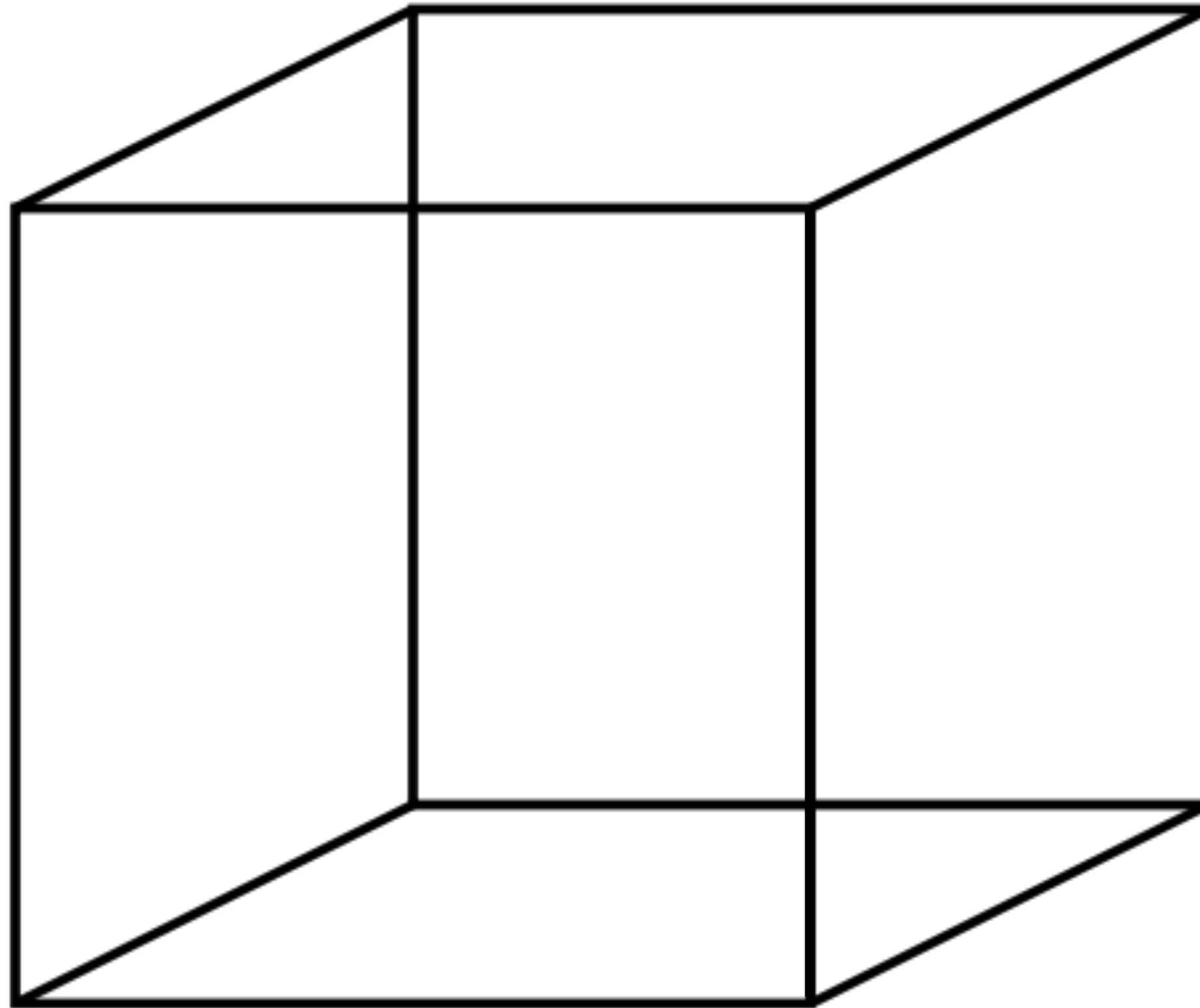


# Higher-dimensional quadrature



**Fubini's theorem:**

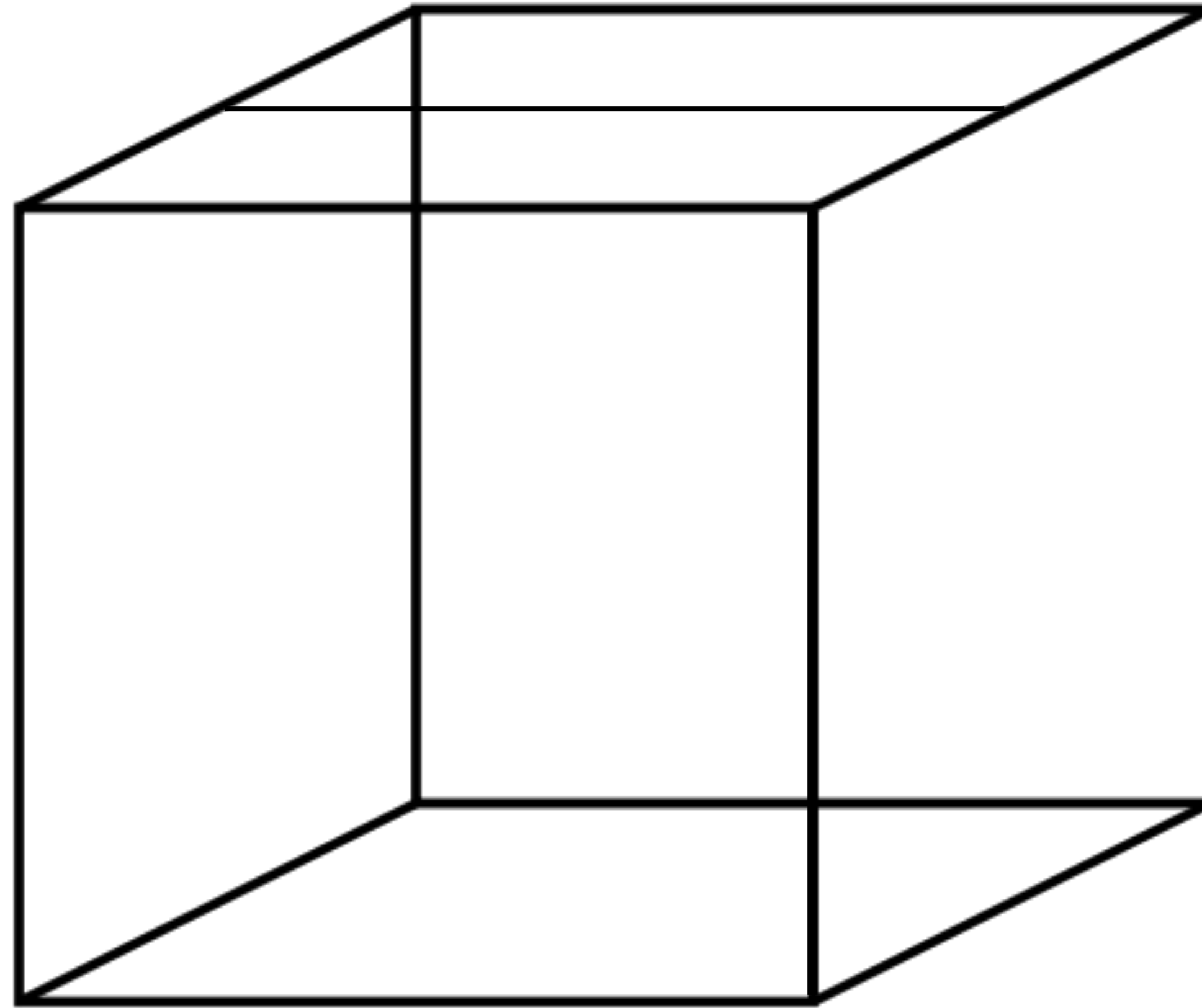
# Higher-dimensional quadrature



**Fubini's theorem:**

$$\int_{X \times Y \times Z} f(x, y, z) \, d(x, y, z) = \int_Z \left( \int_Y \left( \int_X f(x, y, z) \, dx \right) dy \right) dz = \dots$$

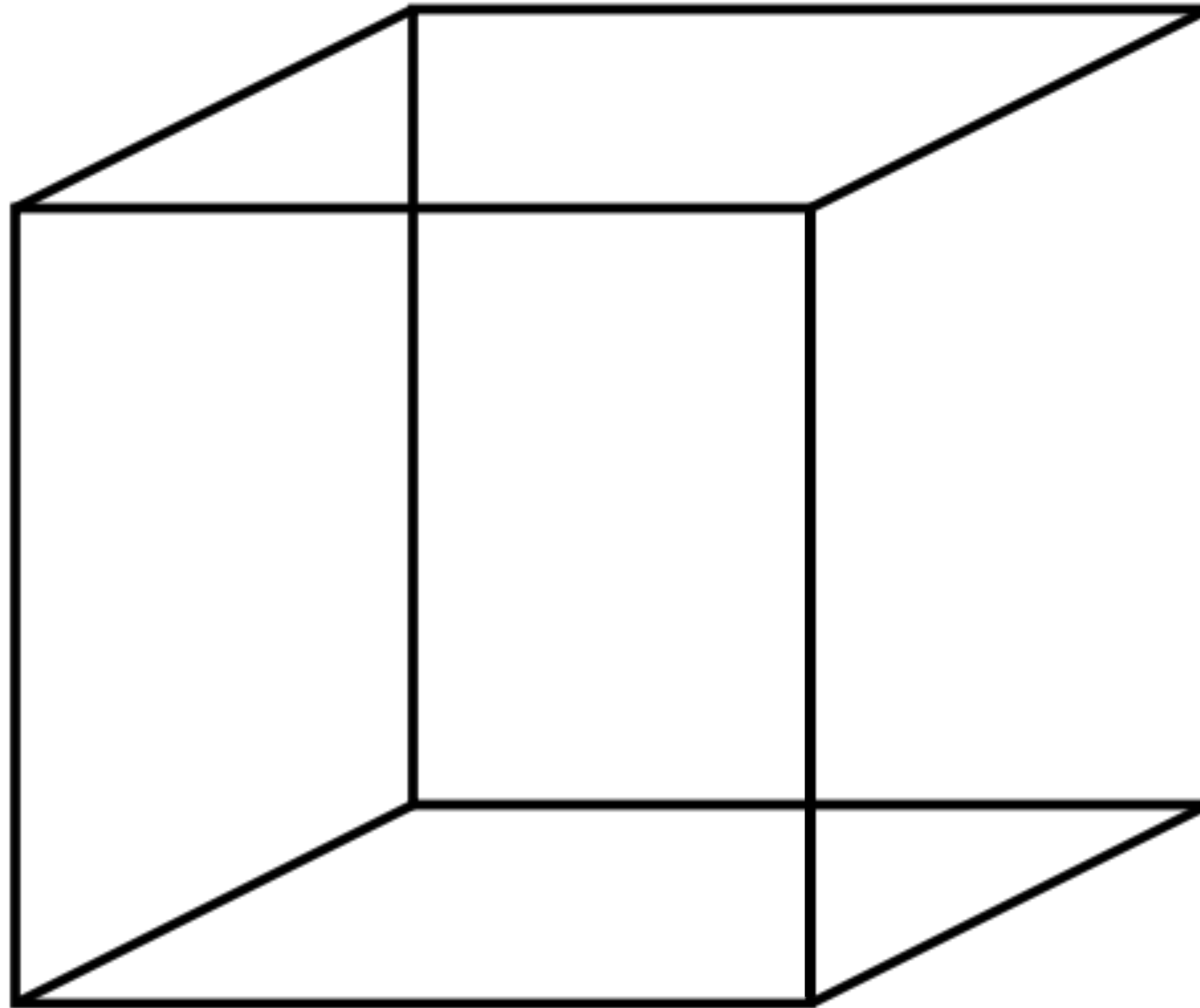
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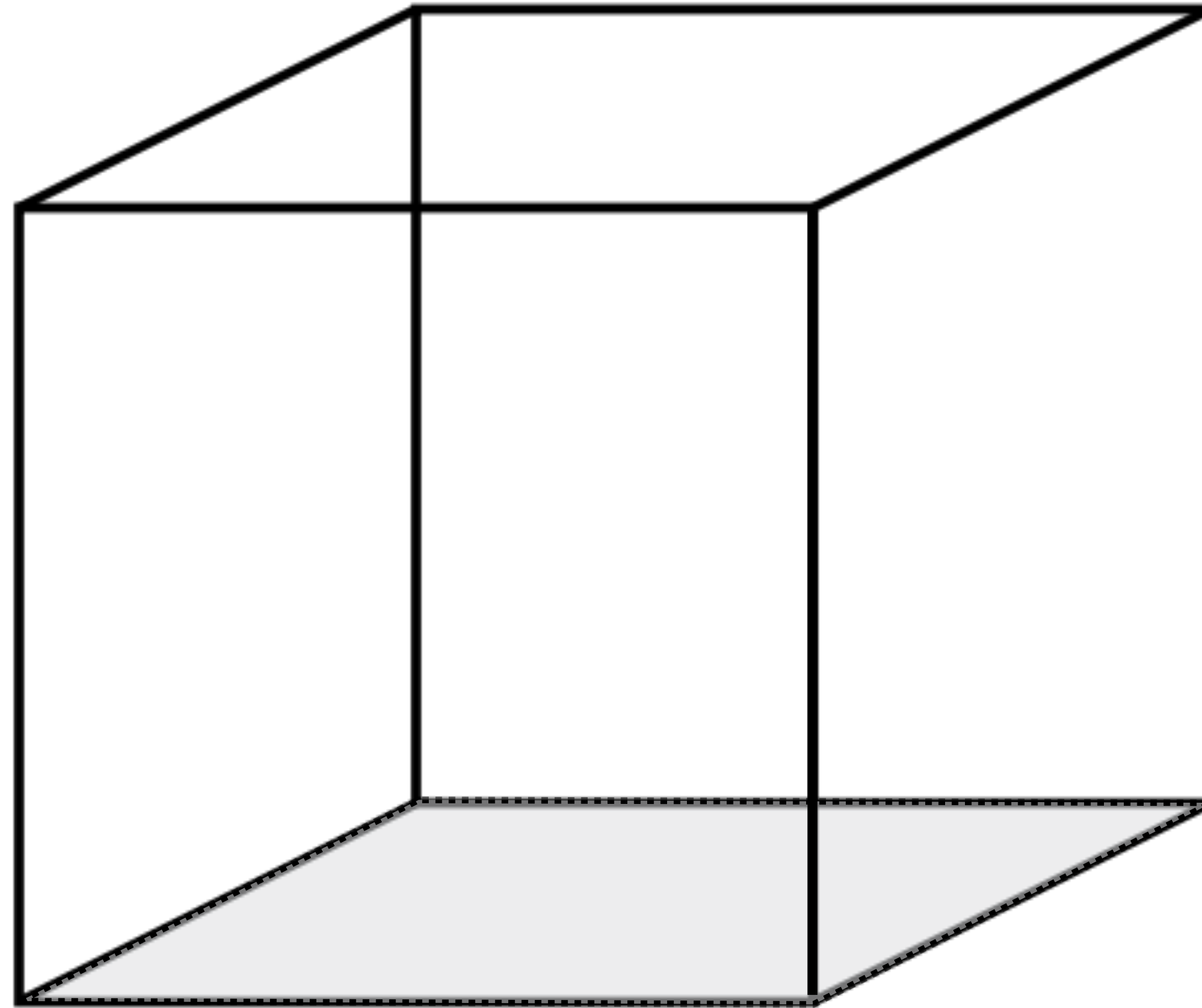
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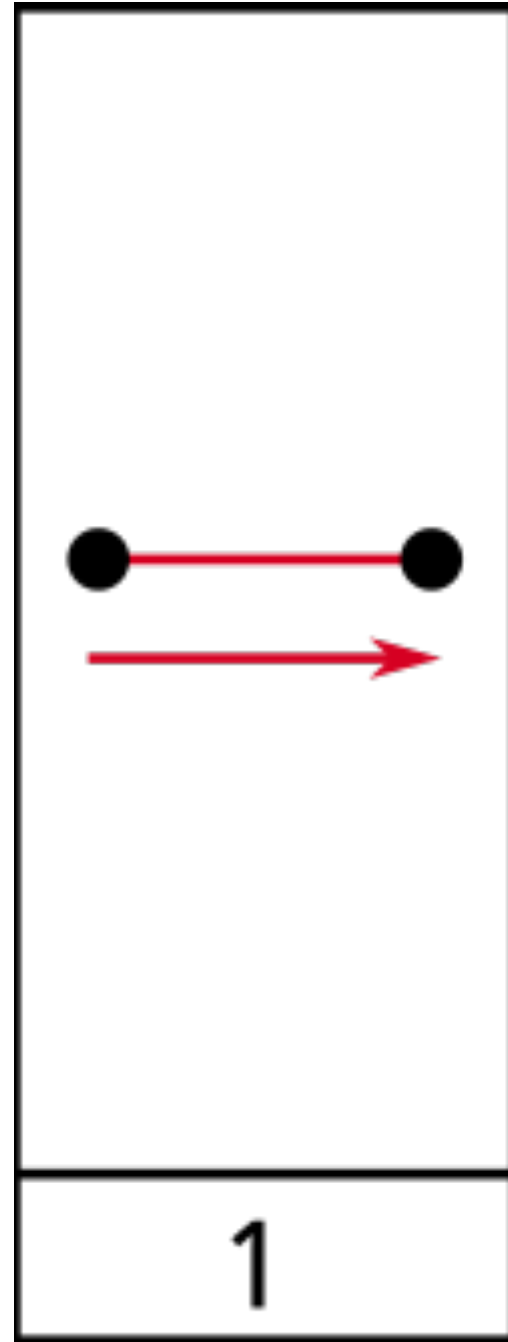


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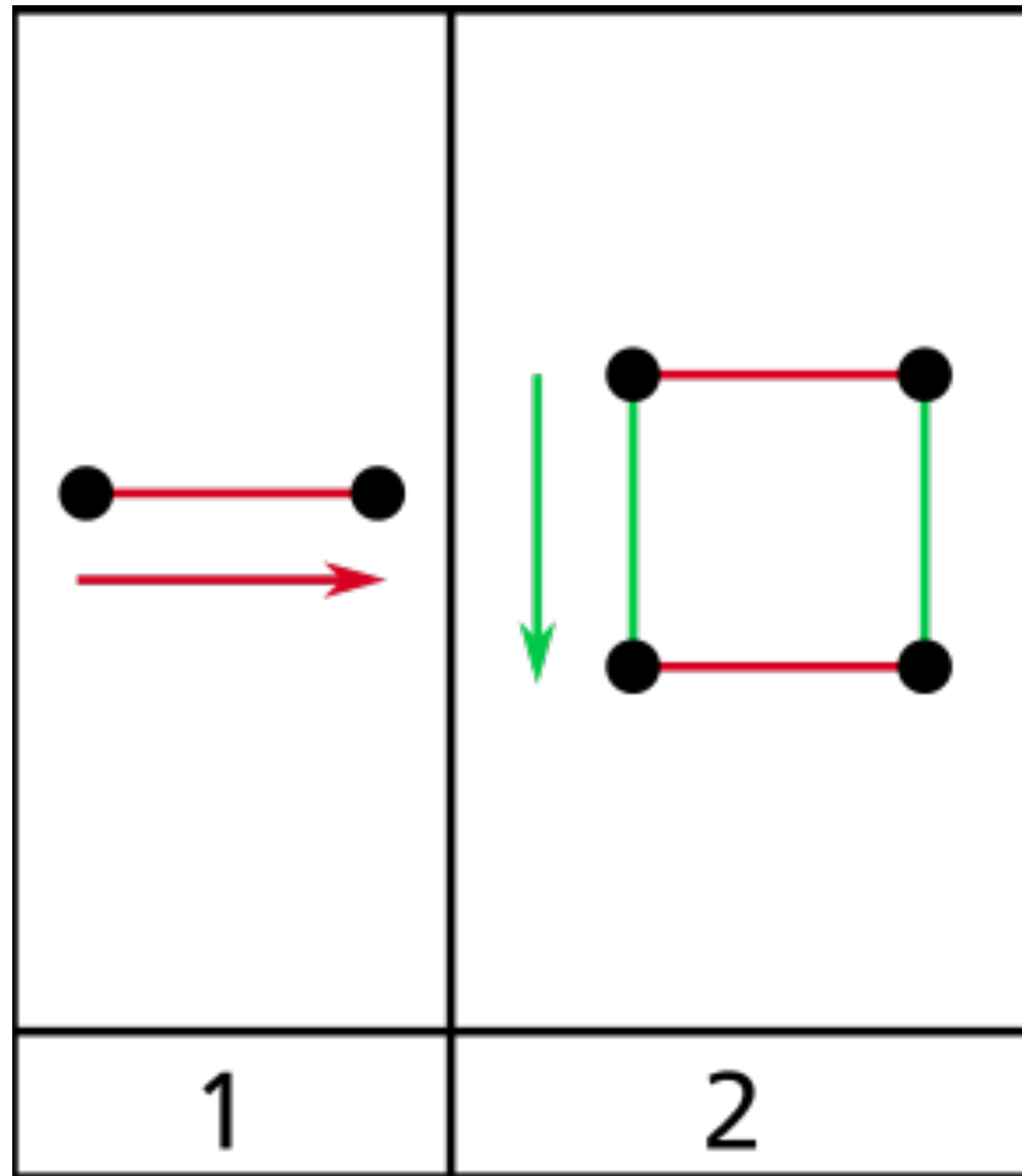
# The curse of dimensionality

Sampling a function on a regular grid becomes completely impractical in higher dimensions.



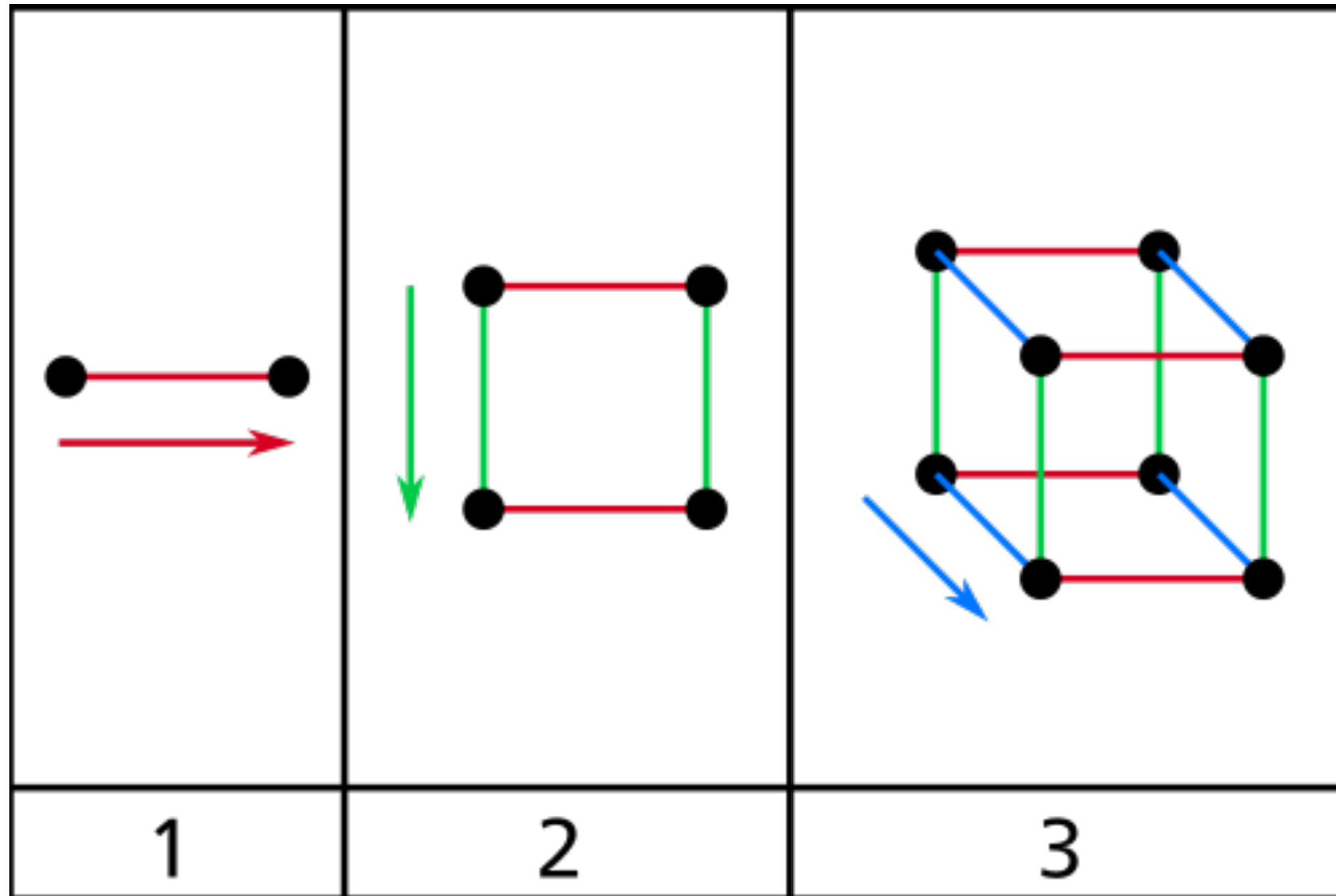
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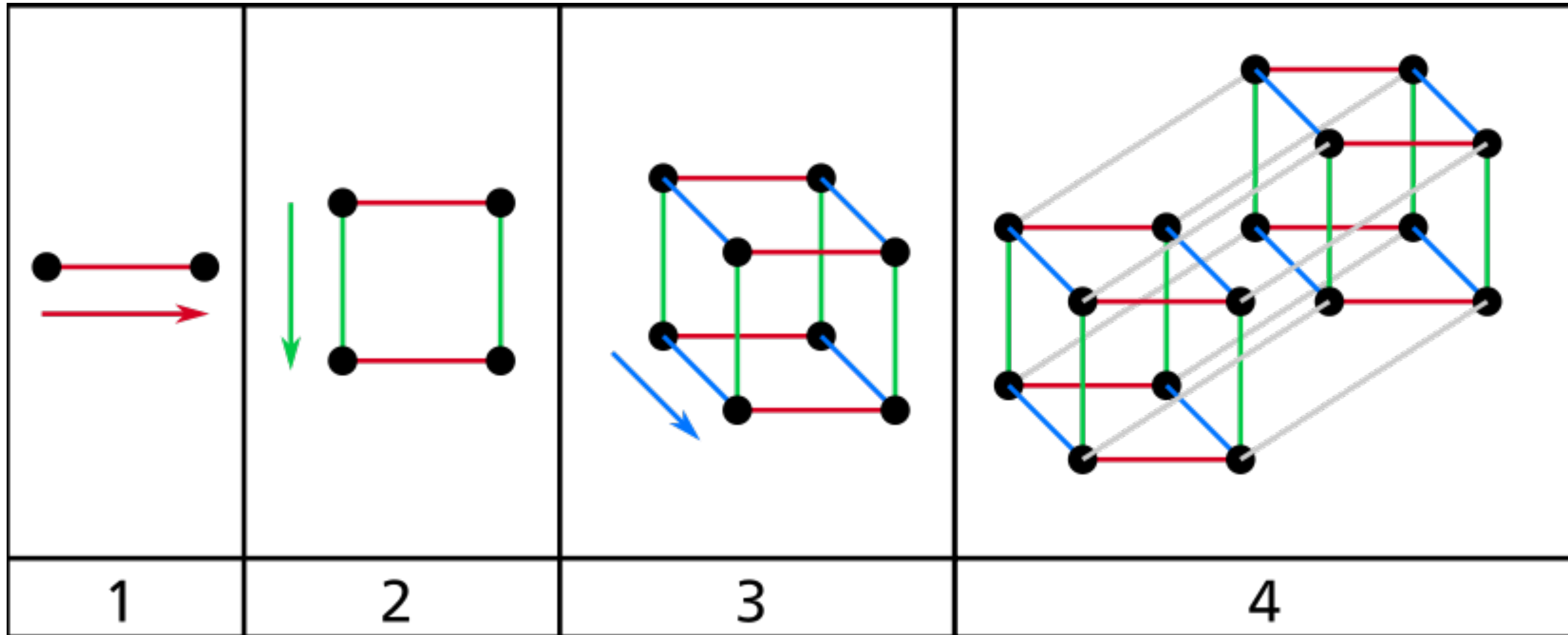
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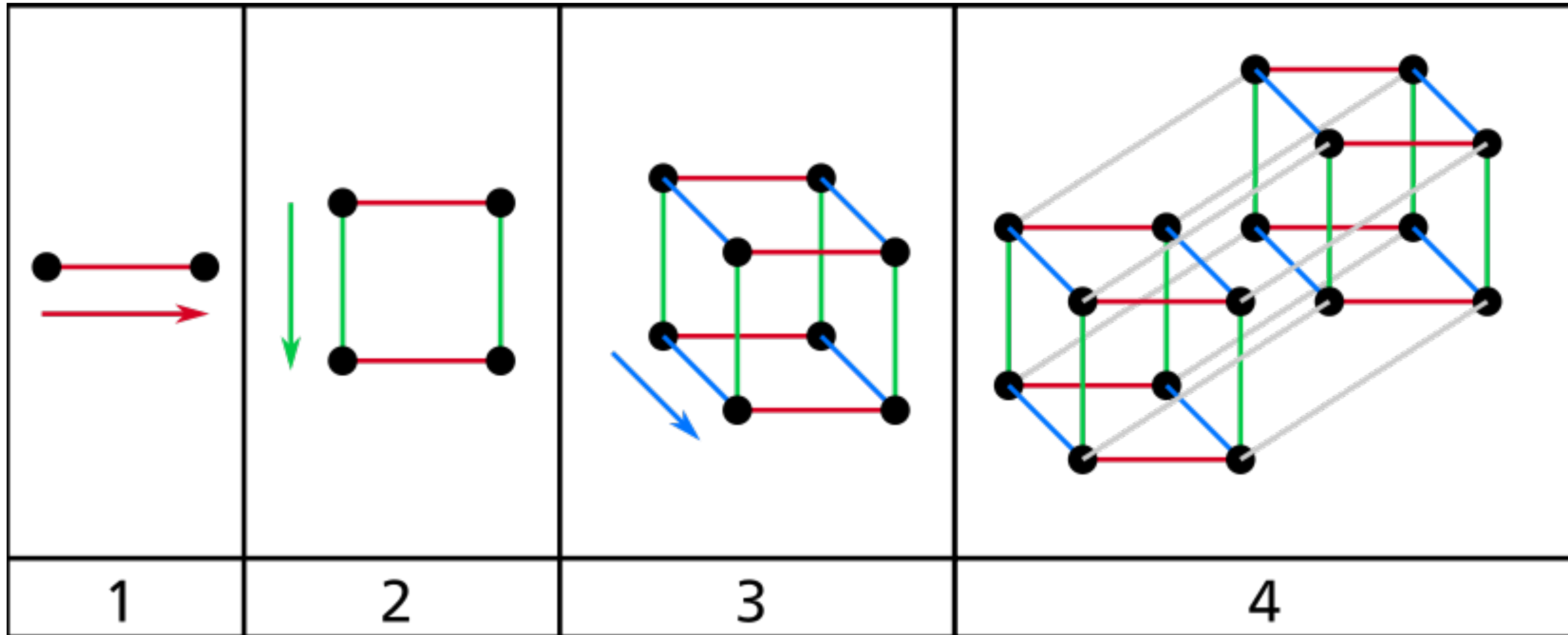
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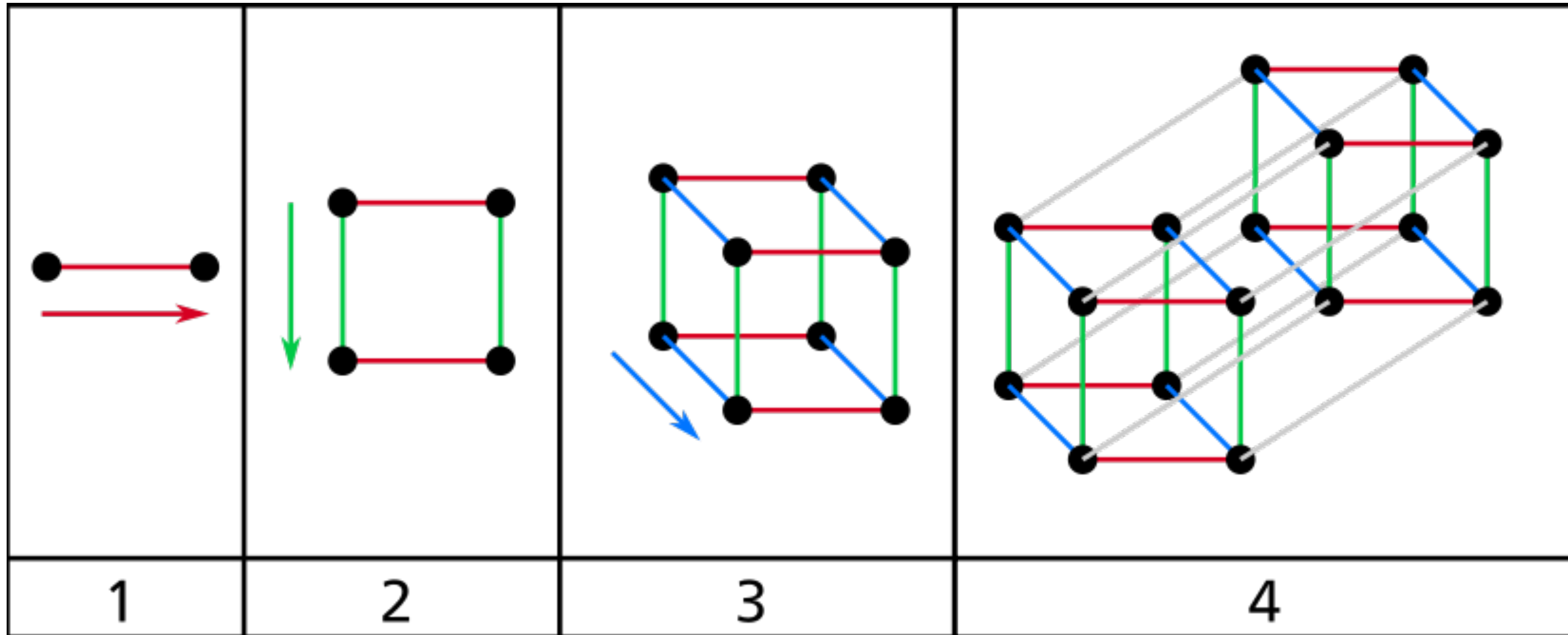
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A 100-dimensional cube has  
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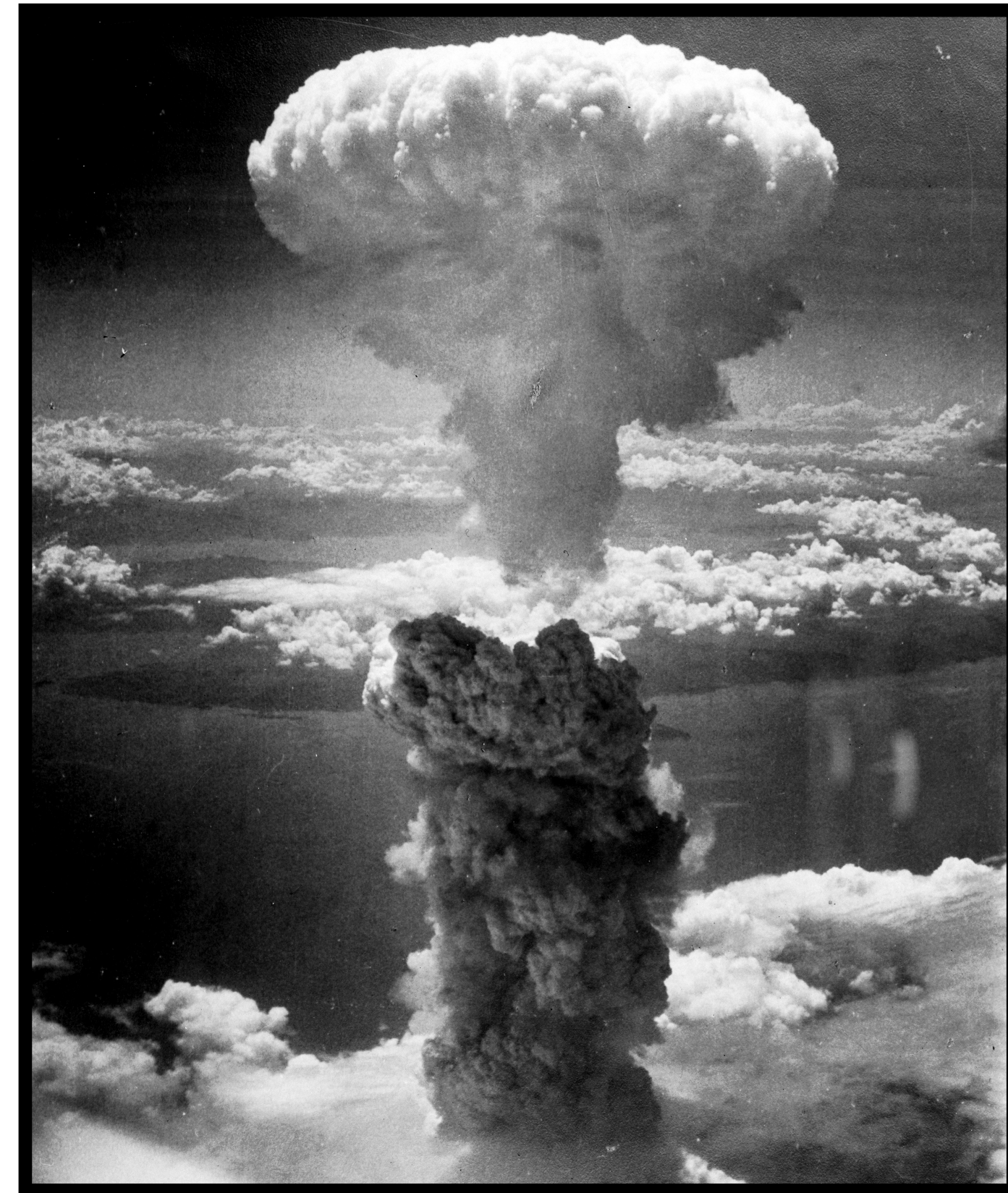
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(!!!)

# Monte Carlo Integration

# History of the Monte Carlo method

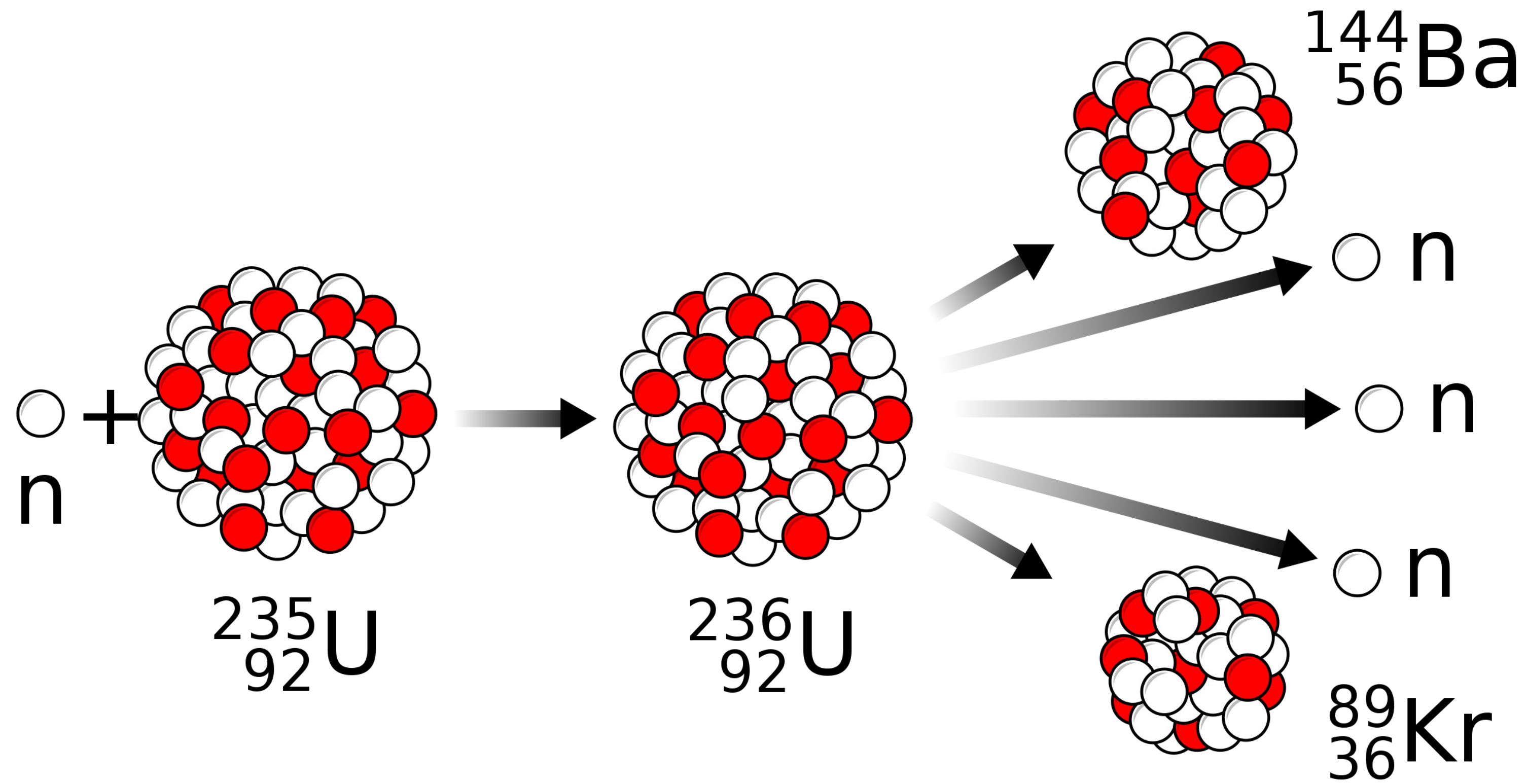
- Developed in the 1940s in the context of the development of the atomic bomb
- General & very easy to implement.
- Uses randomness to simulate many possible realizations of a complex process
- Very slow convergence



# The Manhattan Project



# The Manhattan Project

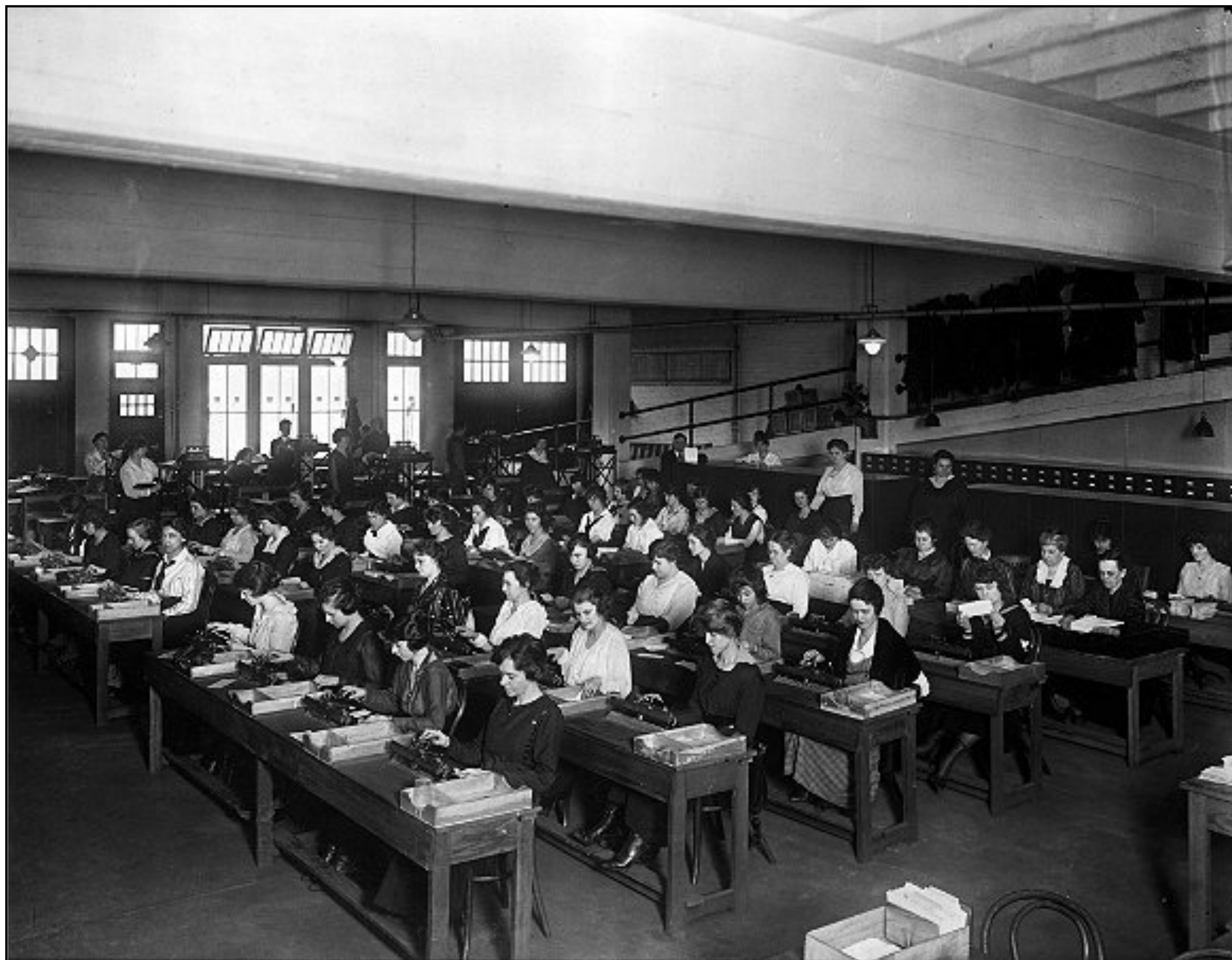


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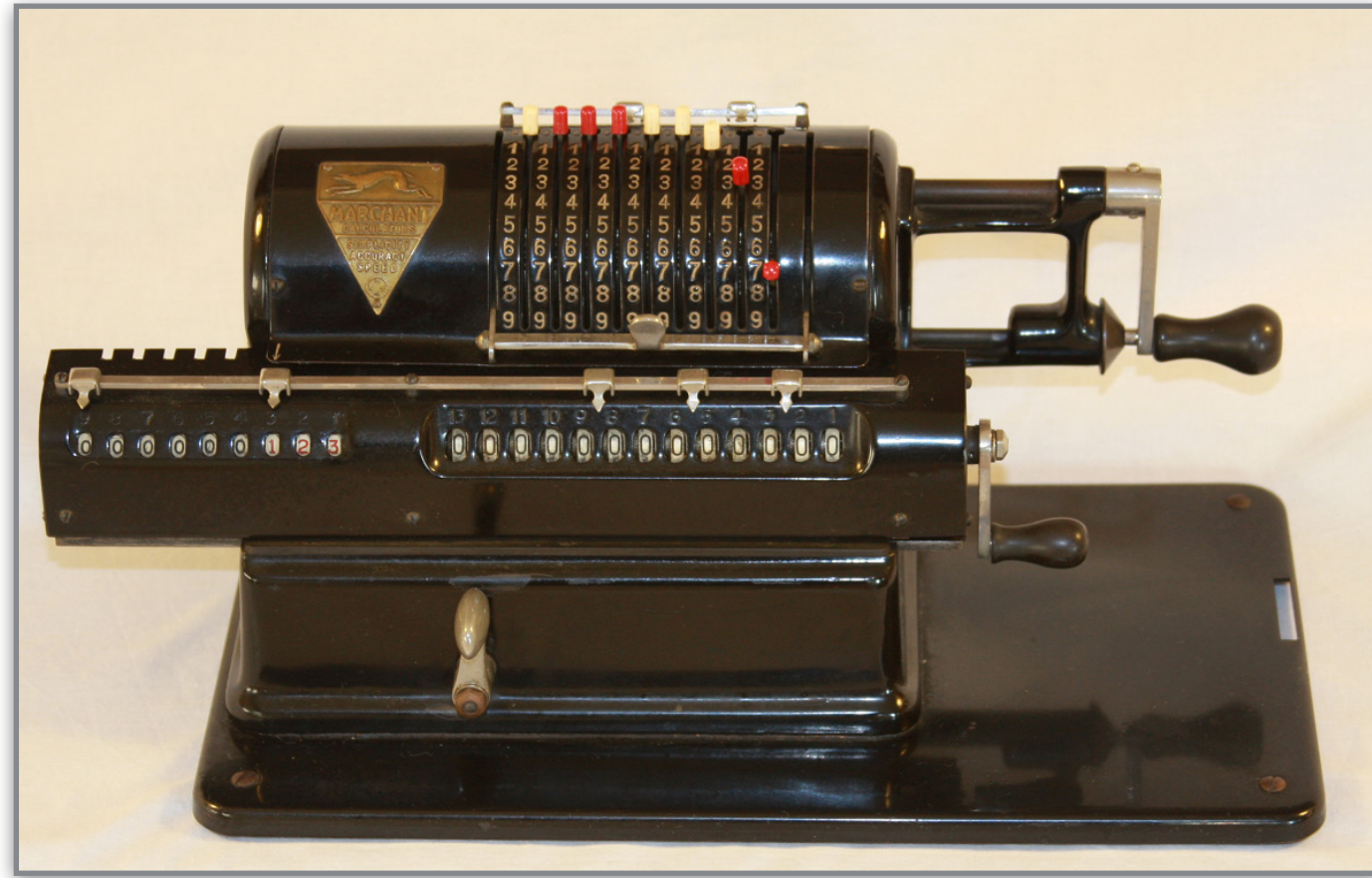
[Godiva device, Wikipedia commons]



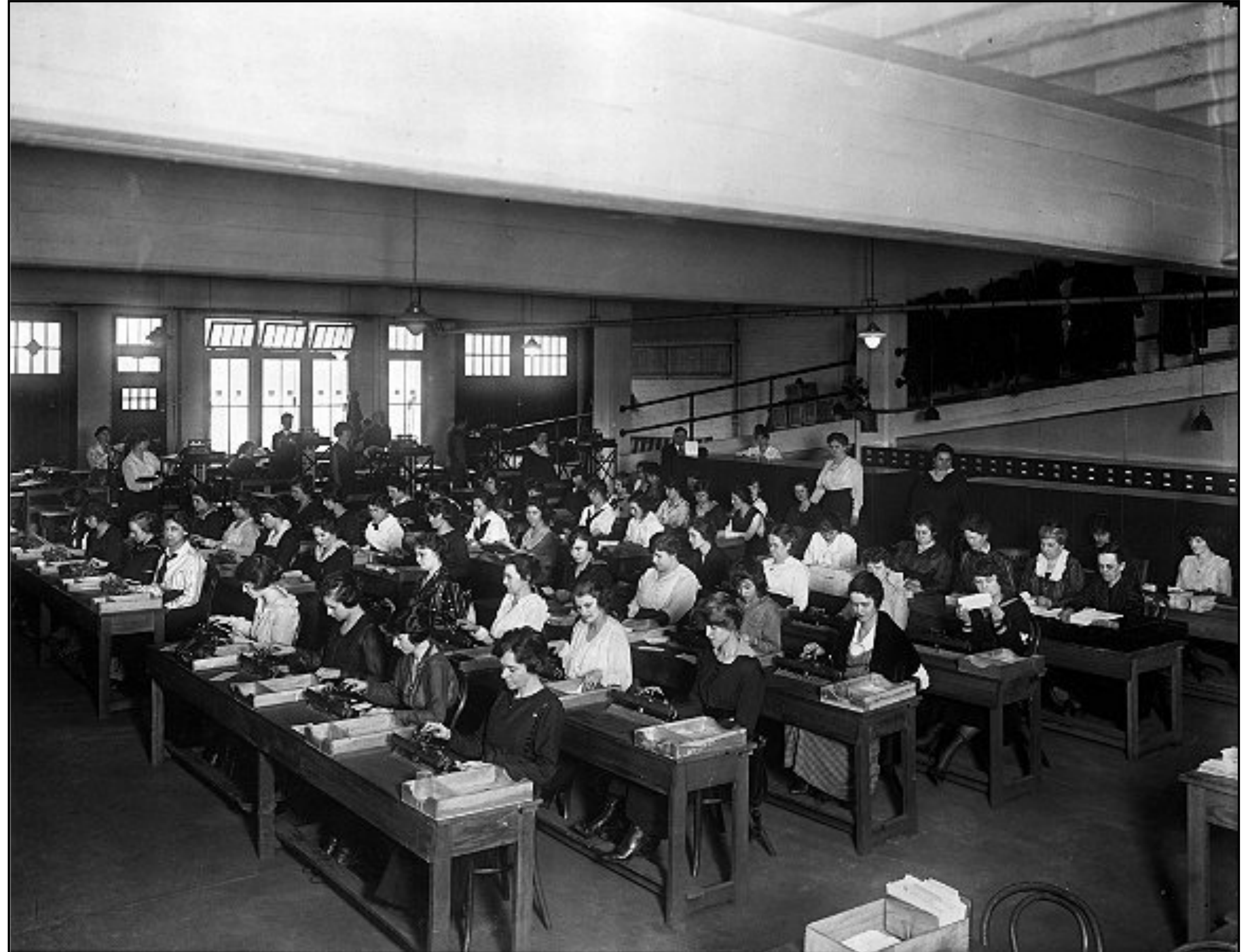
# Human computers



# Human computers



Marchant desk calculator



# Stanisław Ulam's hospital stay



Lose



Win

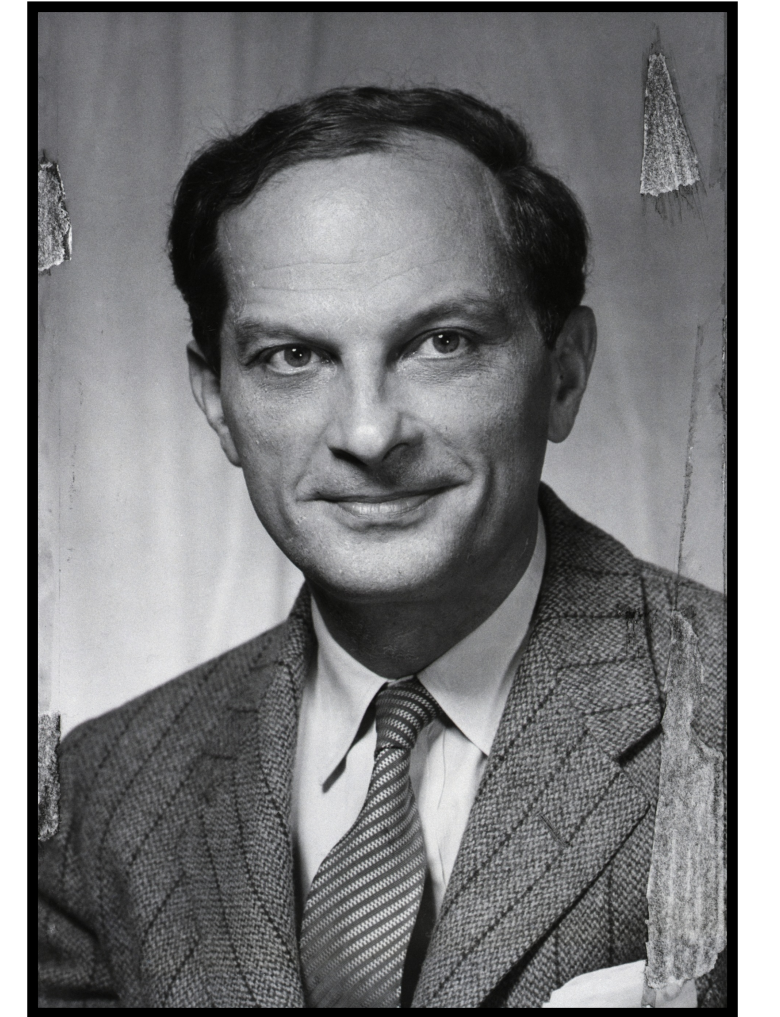


Win



Lose

What's the chance of winning with a properly shuffled deck?



Stanisław Ulam

# Stanisław Ulam's hospital stay



Lose



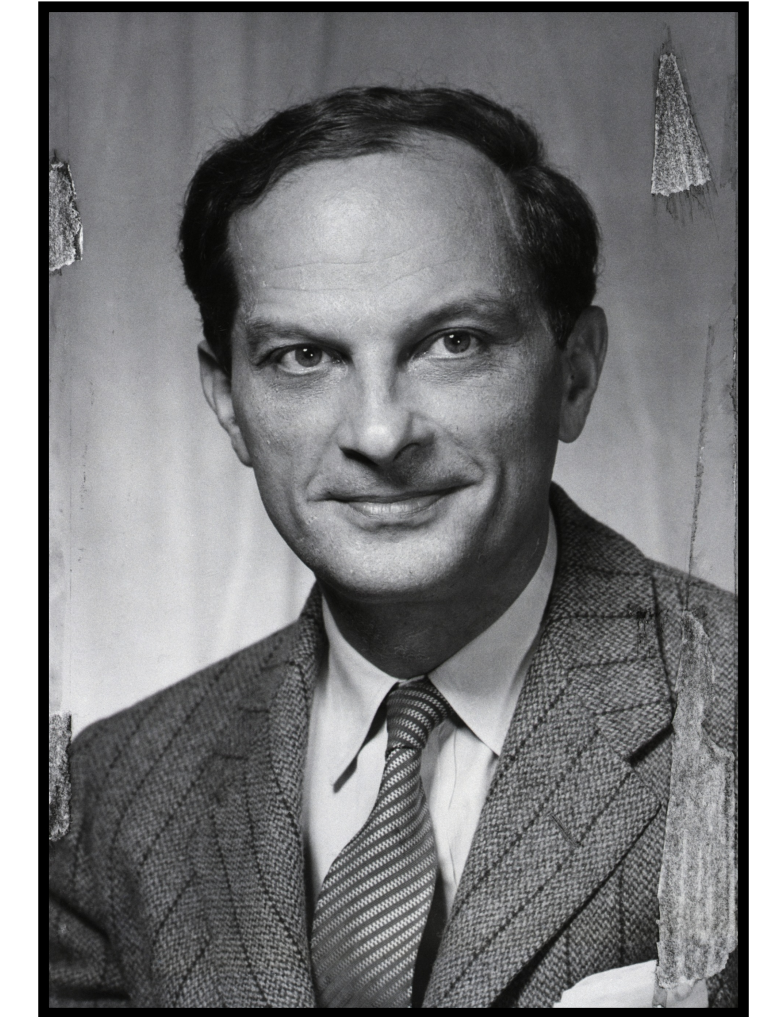
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Stanisław Ulam

What's the chance of winning with a properly shuffled deck?

$$P = \lim_{n \rightarrow \infty} P_n \quad \text{where} \quad P_n = \frac{1}{n} \sum_{i=1}^n \begin{cases} 1, & \text{game } i \text{ is won,} \\ 0, & \text{game } i \text{ is lost} \end{cases}$$

# Recap: Birthday Paradox

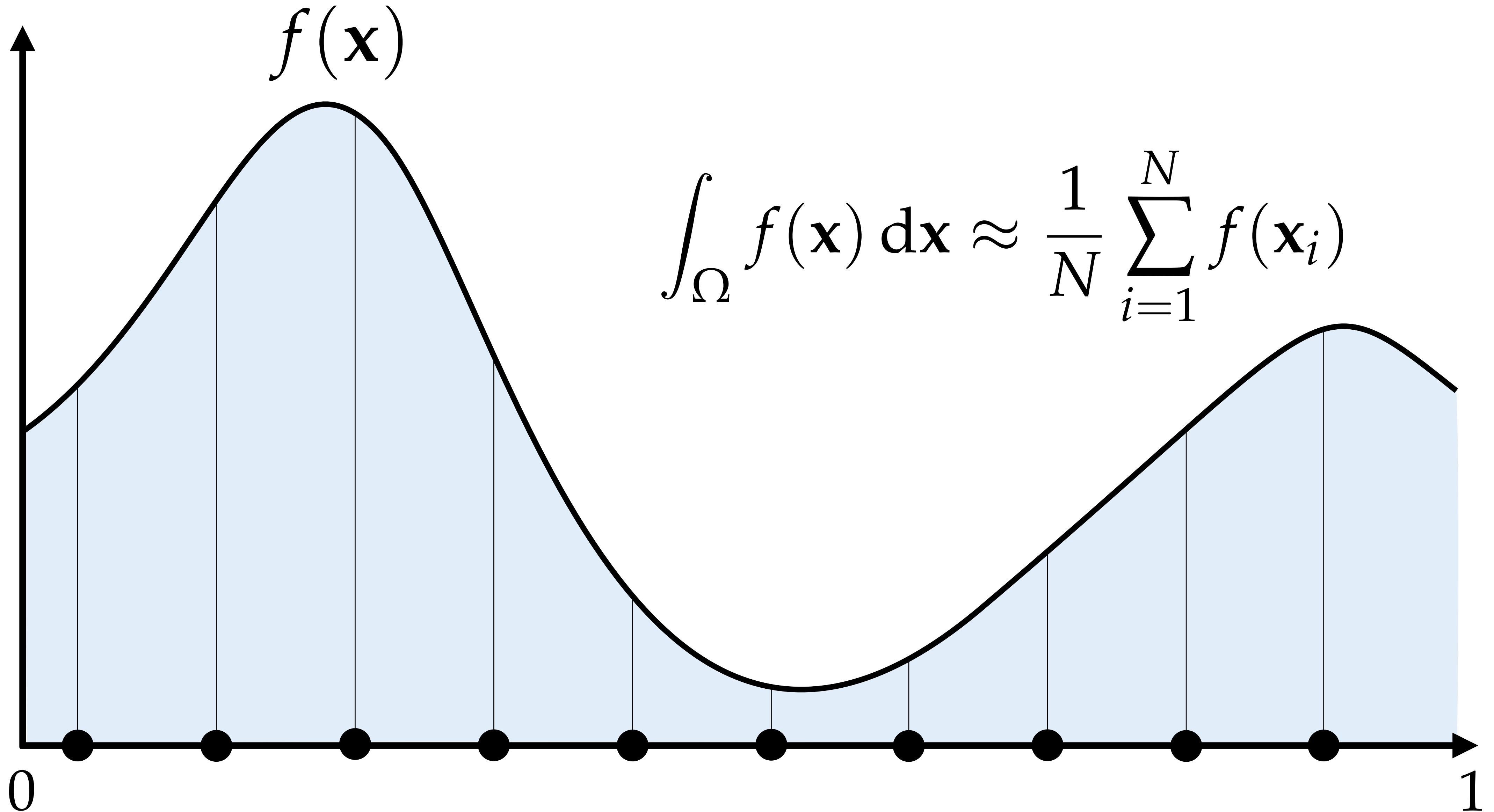
You have **already used** a Monte Carlo method in CS328!

```
In [7]: # For n people, compute an approximation of the probability that two people share the same birthday, using K iterations
def birthday_paradox(n, K):
    s = 0 # Keep track of how often the statement is true

    # Perform K iterations of the same experiment.
    for it in range(K):
        dates = [rnd.randint(0, 365) for i in range(n)]
        if len([x for x in dates if dates.count(x) > 1]) > 0:
            s += 1

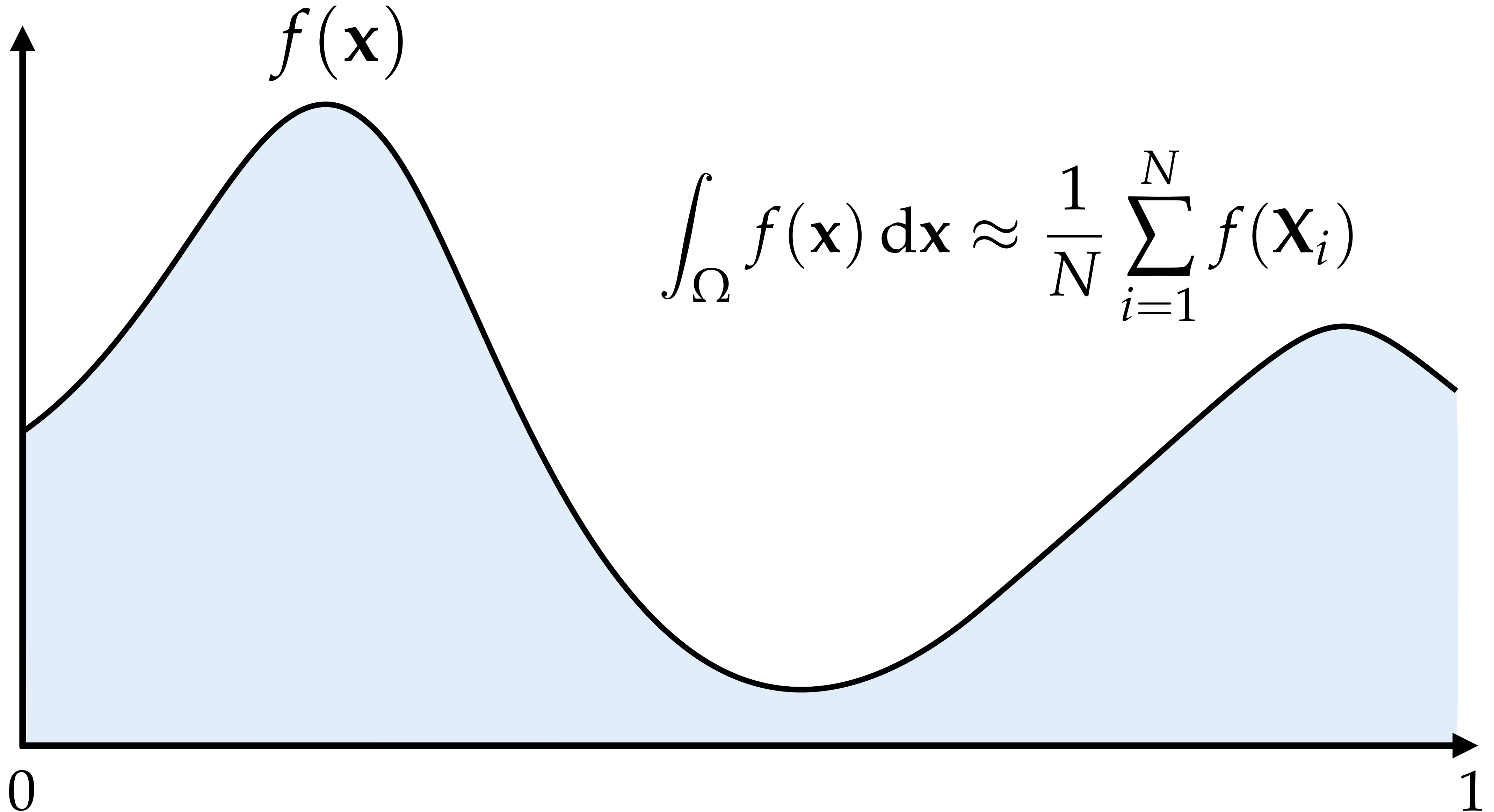
    # Return average probability
    return s / K
```

# Recap: Riemann sums & quadrature

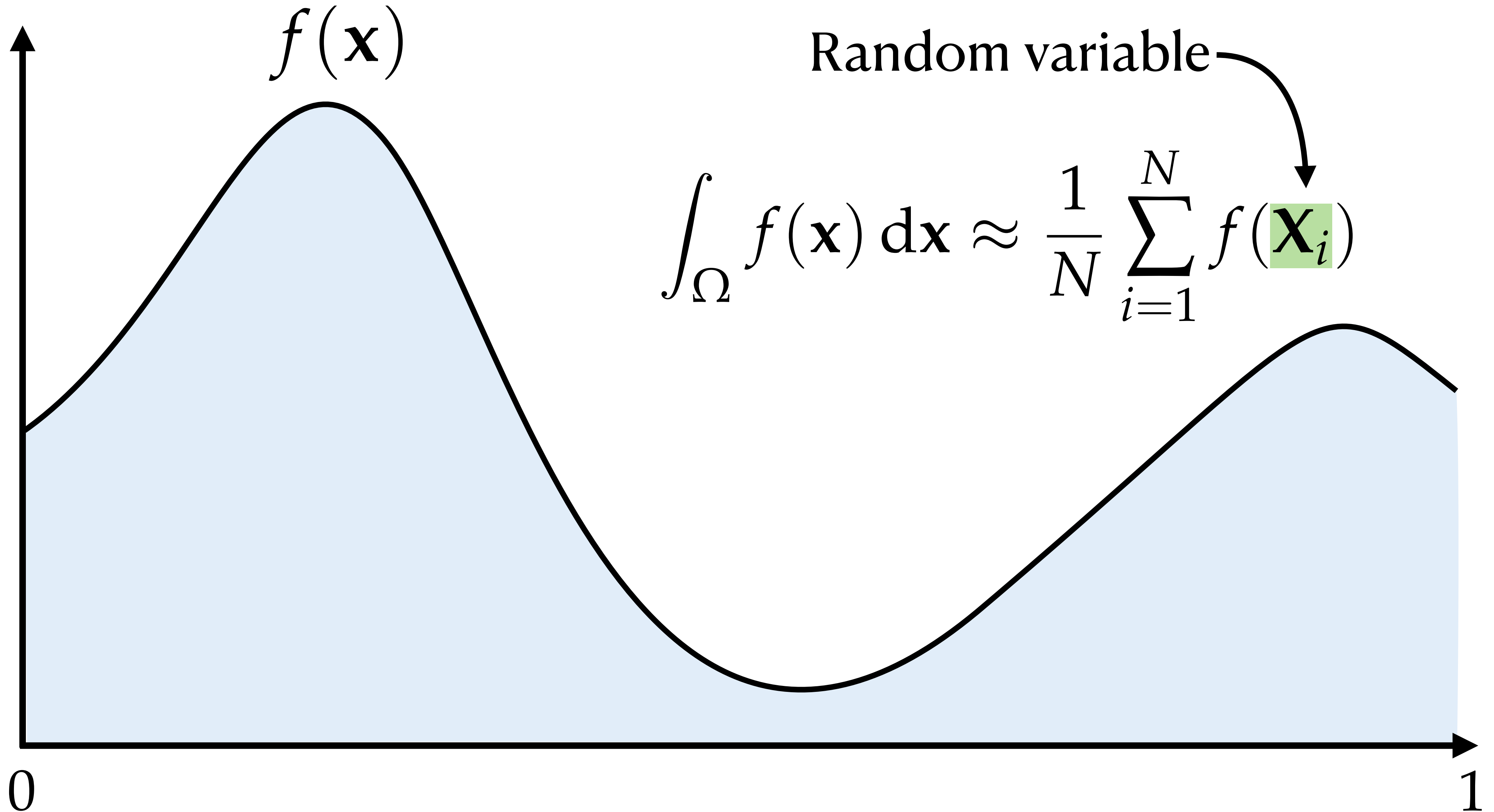


$$\int_{\Omega} f(\mathbf{x}) \, d\mathbf{x} \approx \frac{1}{N} \sum_{i=1}^N f(\mathbf{x}_i)$$

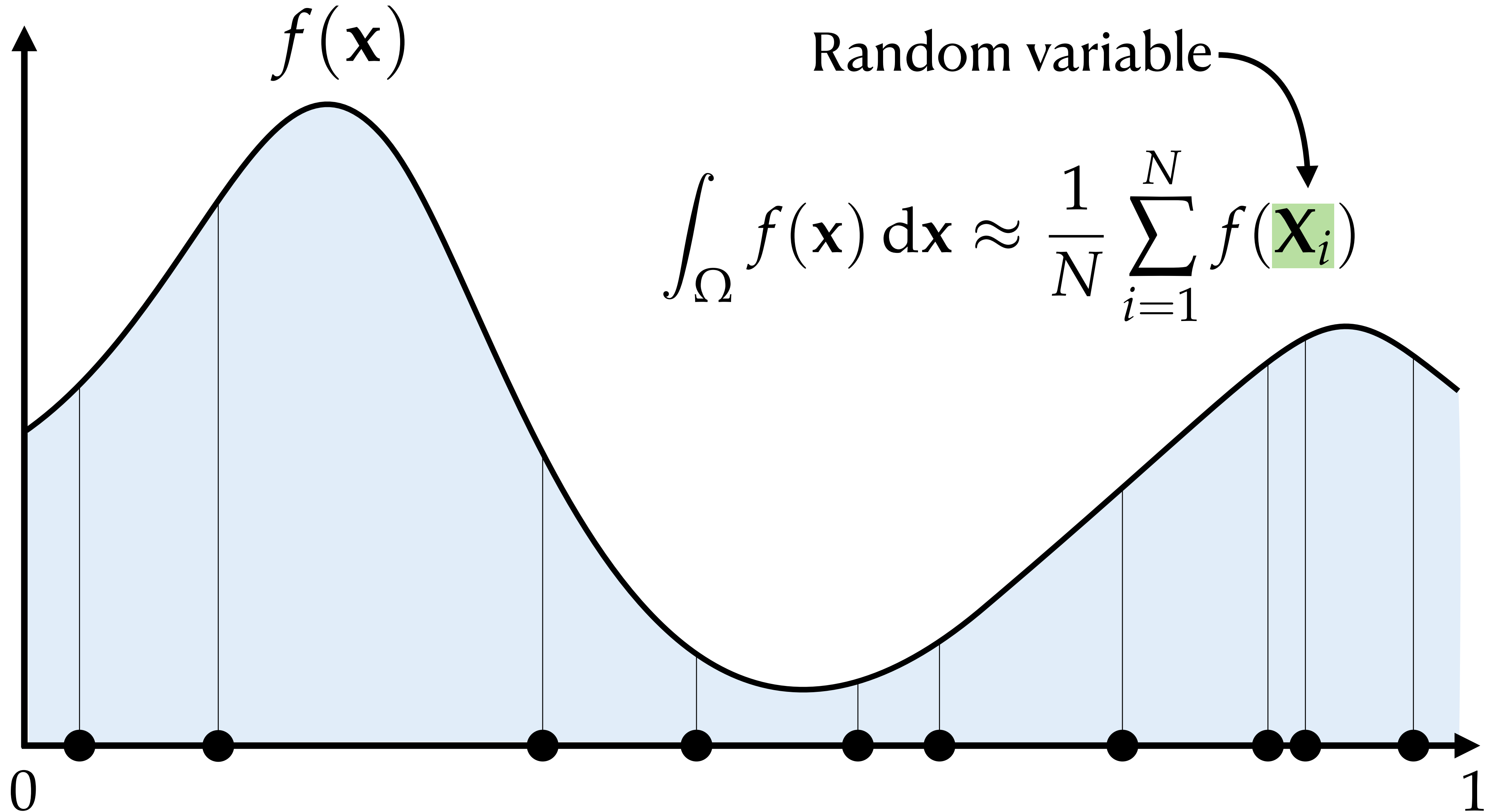
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# Does Monte Carlo integration work?

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Expectation over the continuous set  $\Omega$  of possible outcomes

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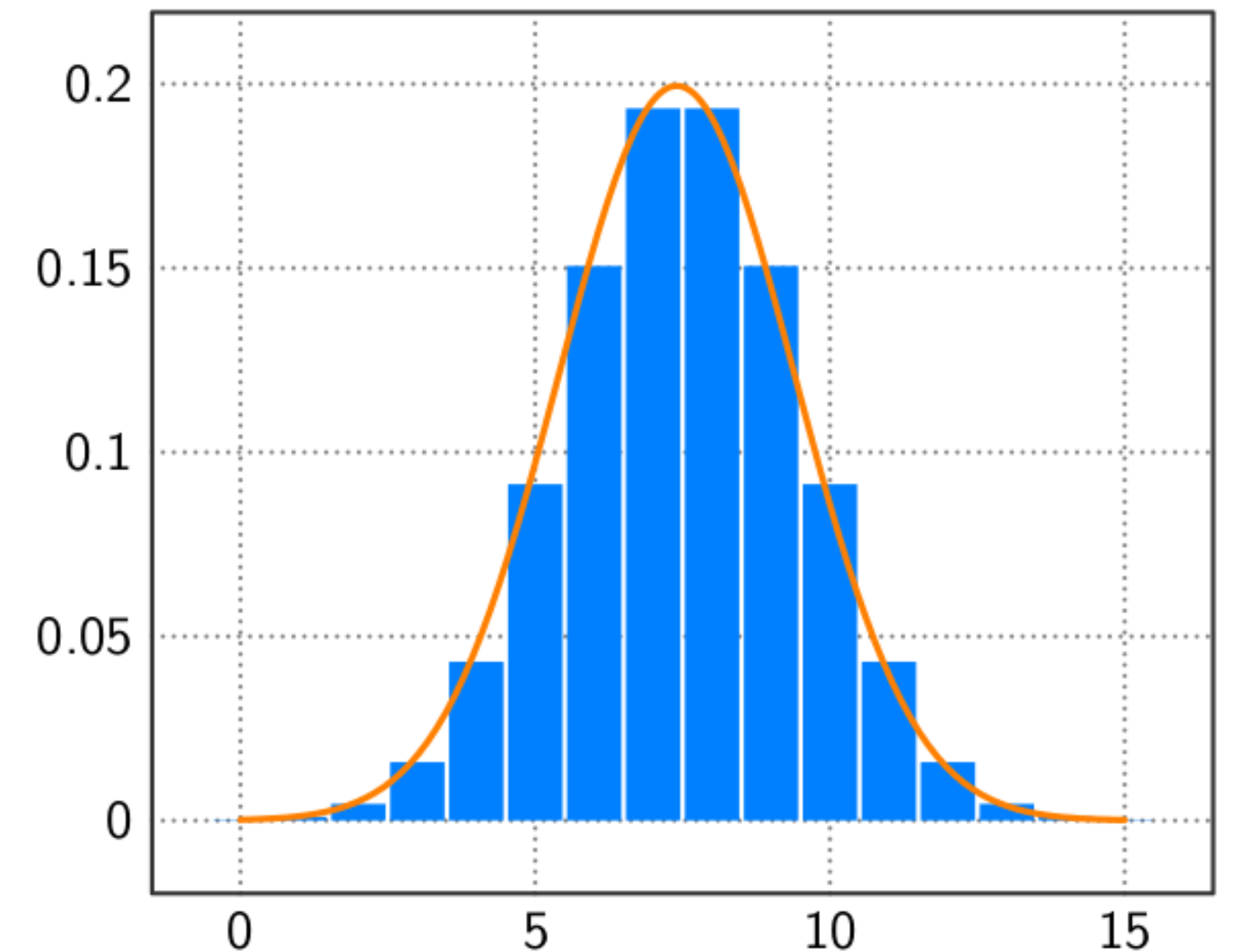
Uniform distribution

# Characteristics

- Result obtained by averaging many samples

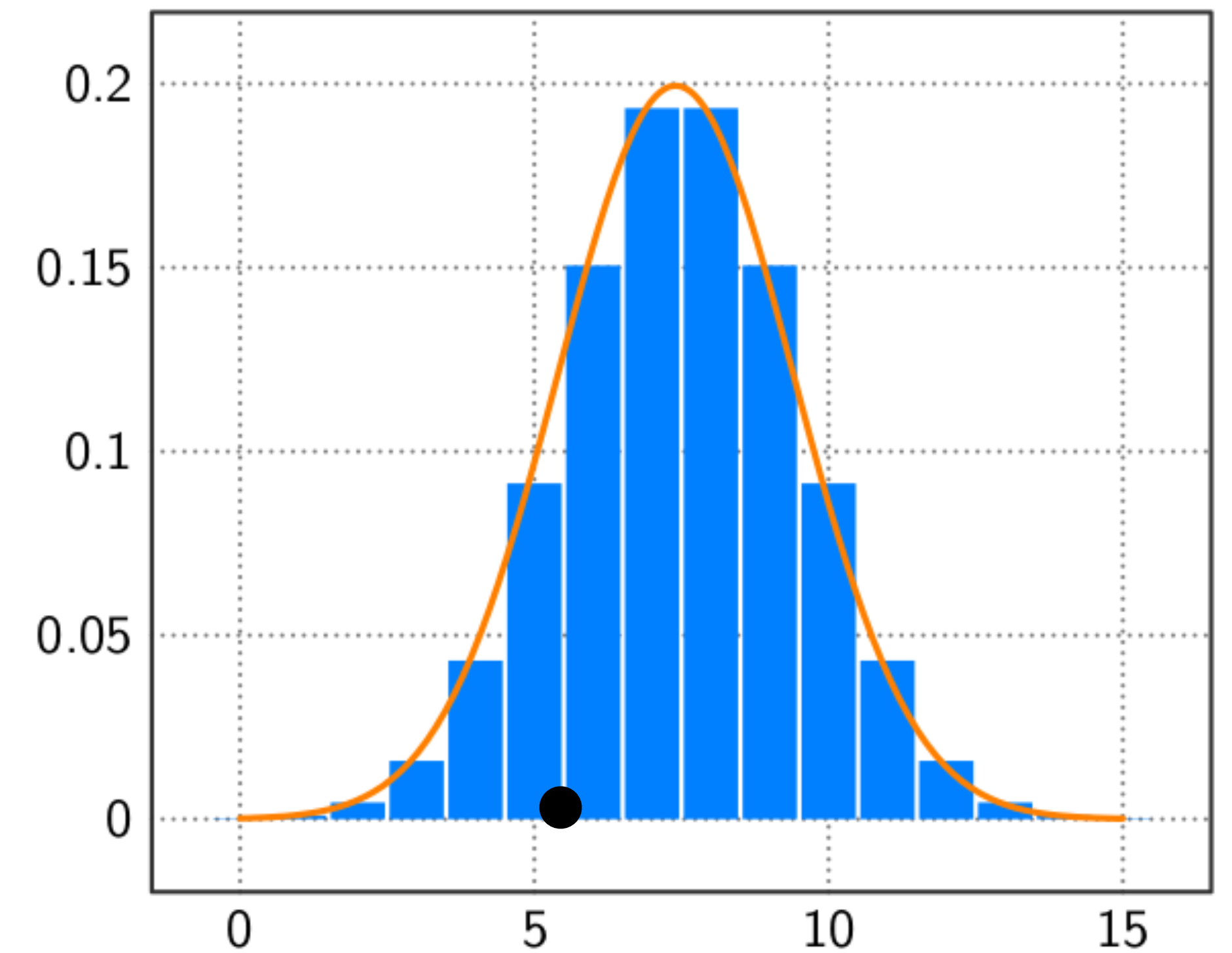
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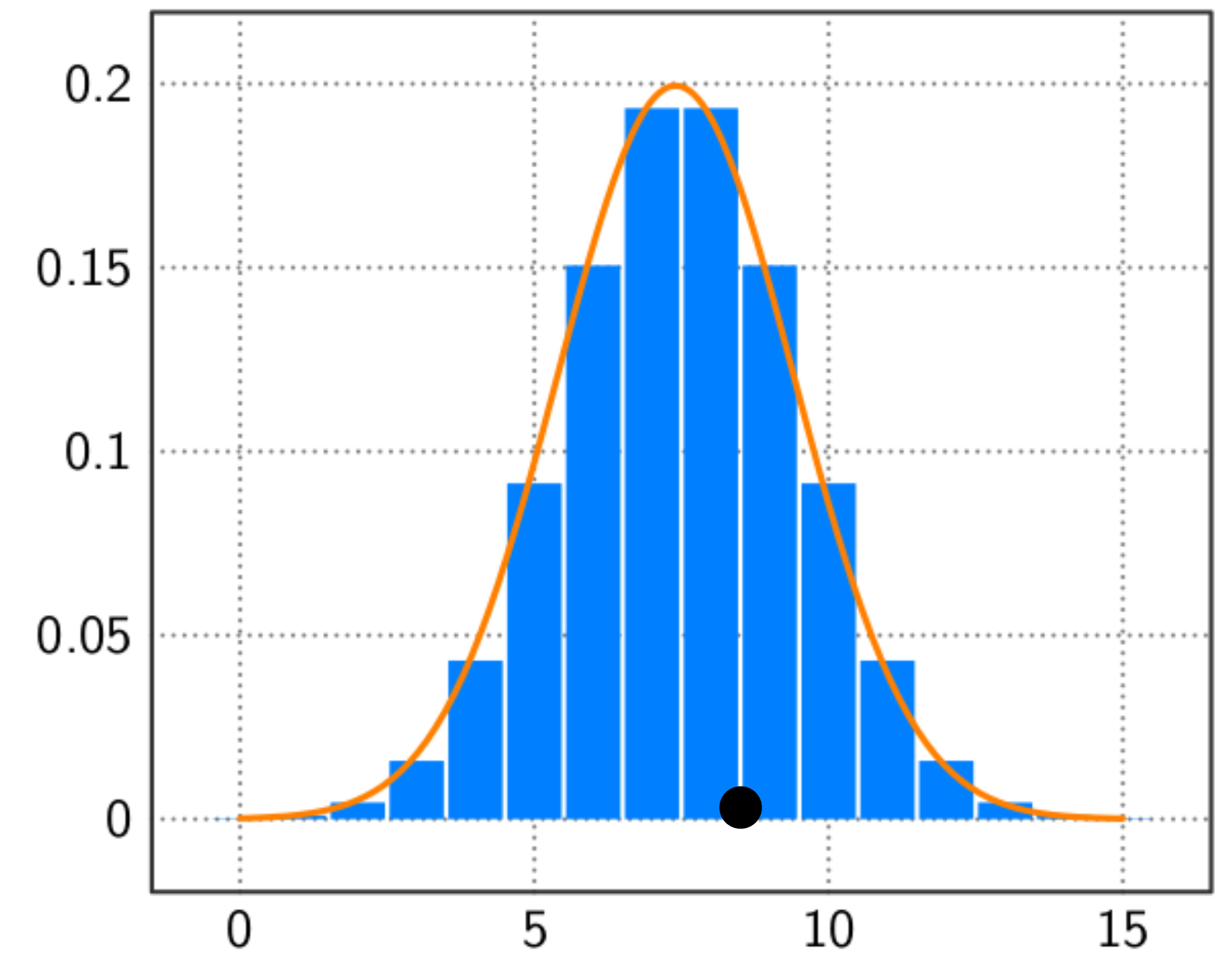
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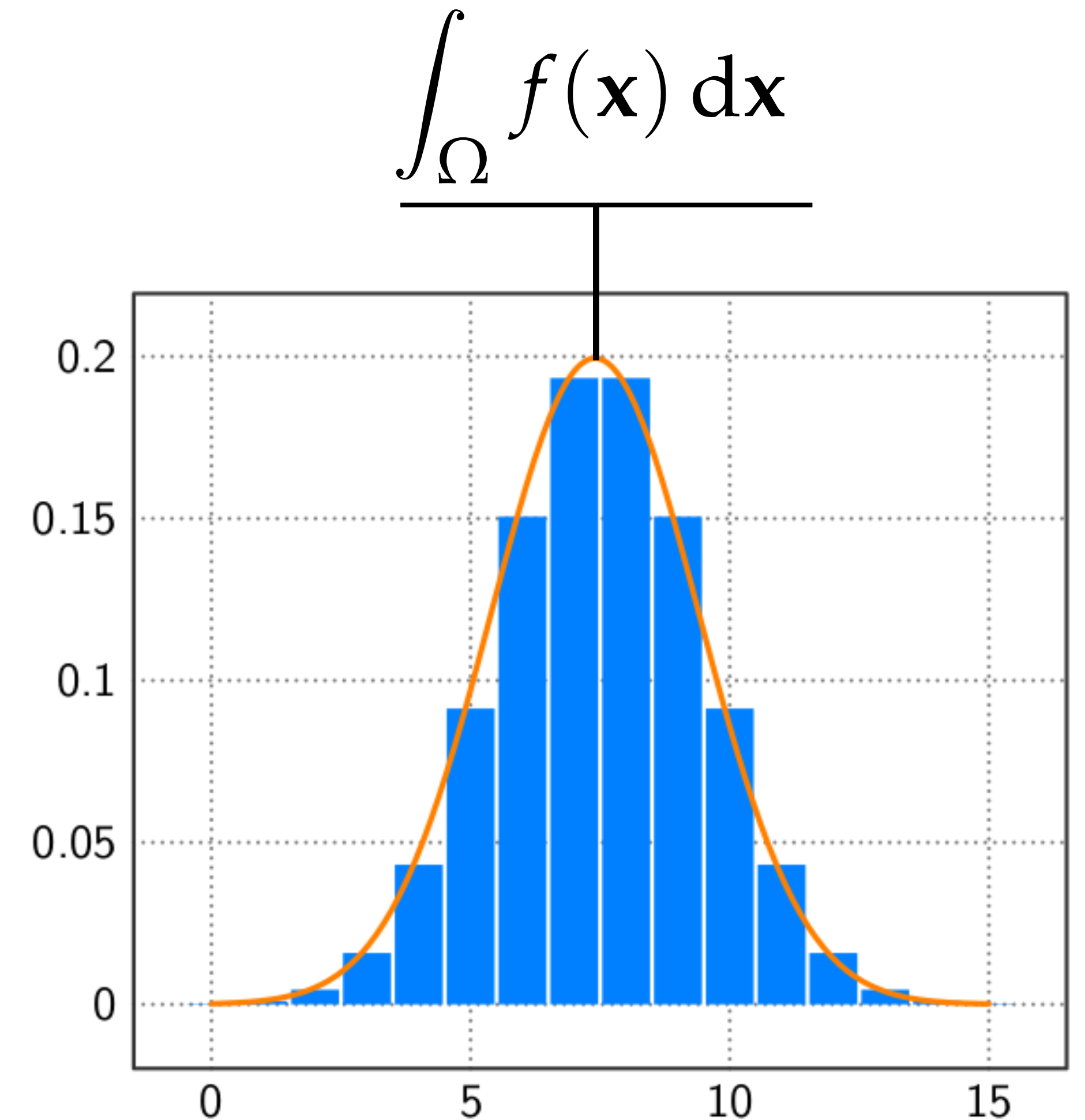
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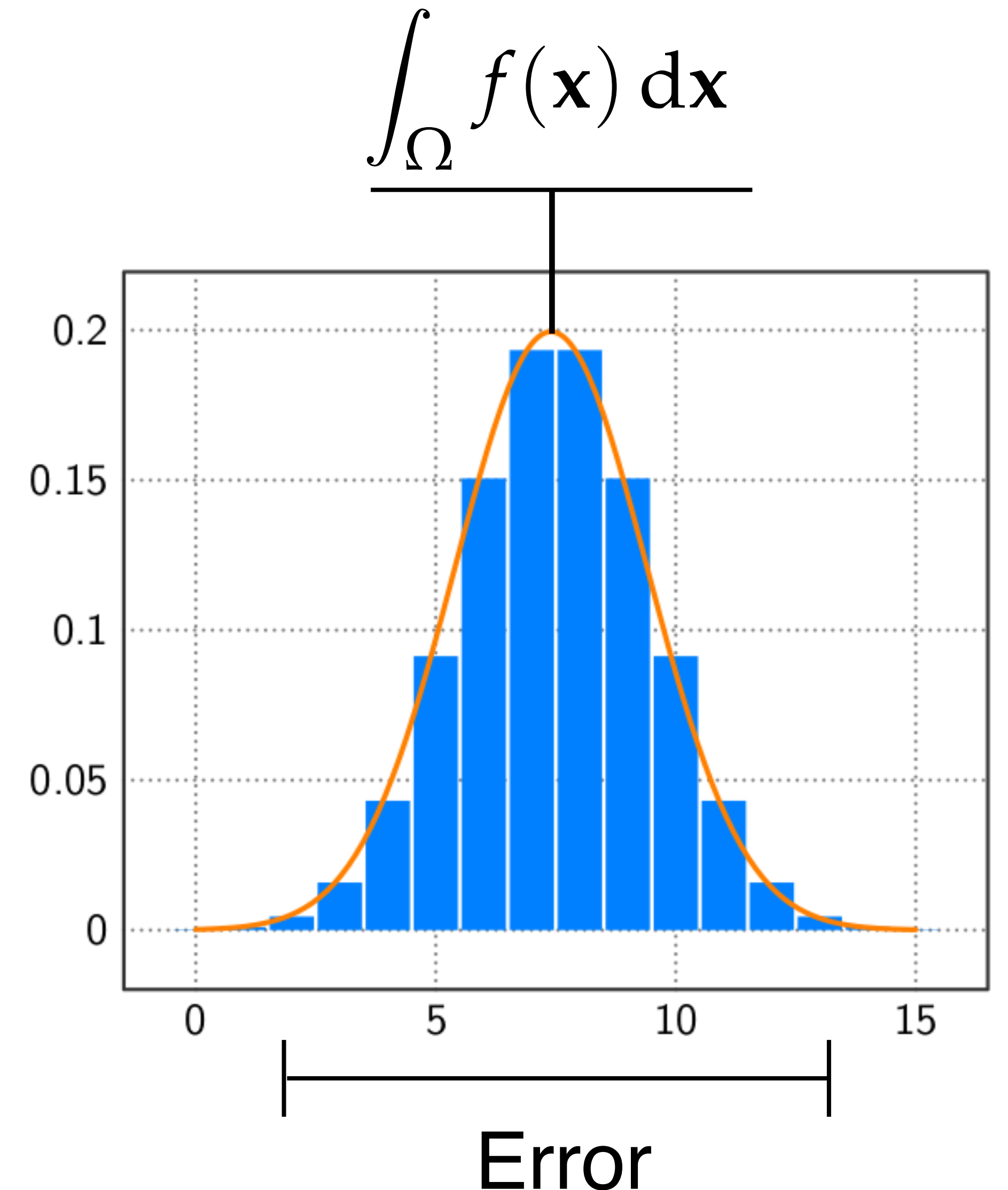
# Characteristics

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# Characteristics

- Result obtained by averaging many samples
- **Central limit theorem** says: the result will tend to a normal distribution.
- Previous slide showed that the distribution is centered around the correct answer.
- **But:** algorithm will give a random answer, so there is always some error. This can be characterized using the variance.



# How well does it work?

Assuming that the random variables  $\mathbf{X}_i$  are independent:

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Dimension doesn't influence  
is the square root of the variance!

the convergence rate!

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# A word of caution

*Monte Carlo is an extremely bad method. It should be used only when all alternative methods are worse.*

*- Alan Sokal*

**Rough guideline:**

1-3 dimensions: quadrature & adaptive quadrature

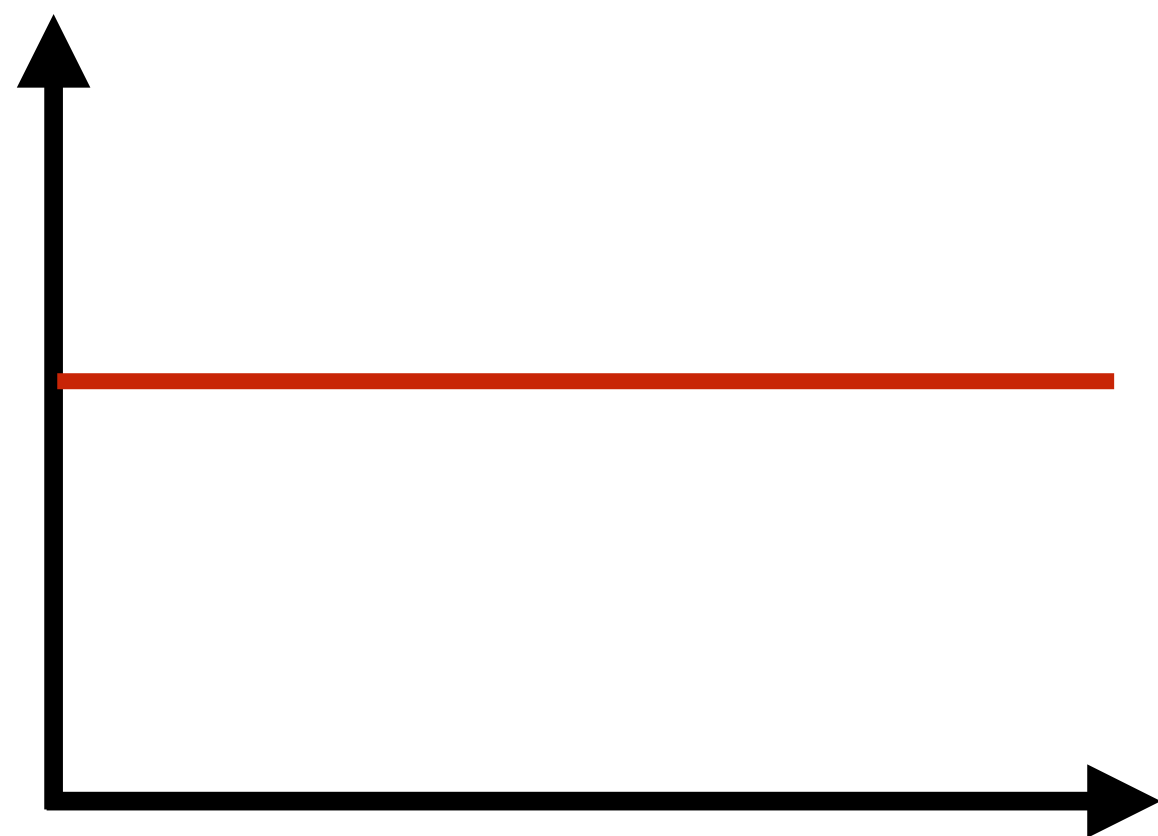
4 dimensions+: Monte Carlo integration

**Demo time**

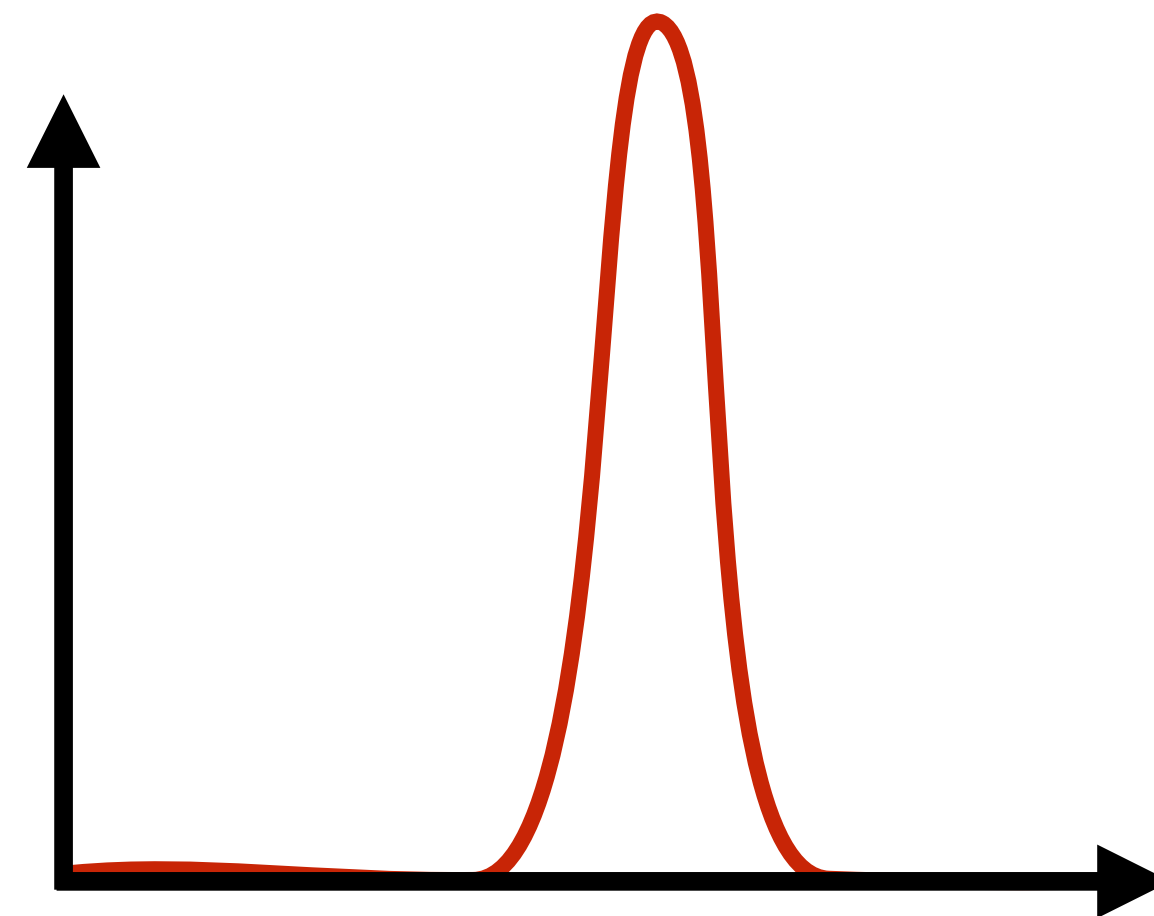
# How much error do we expect?

$$\text{Error} \approx \sqrt{\frac{1}{N} \text{Var}[f(\mathbf{X}_i)]}$$

How big is  $\text{Var}[f(\mathbf{X}_i)]$  ?



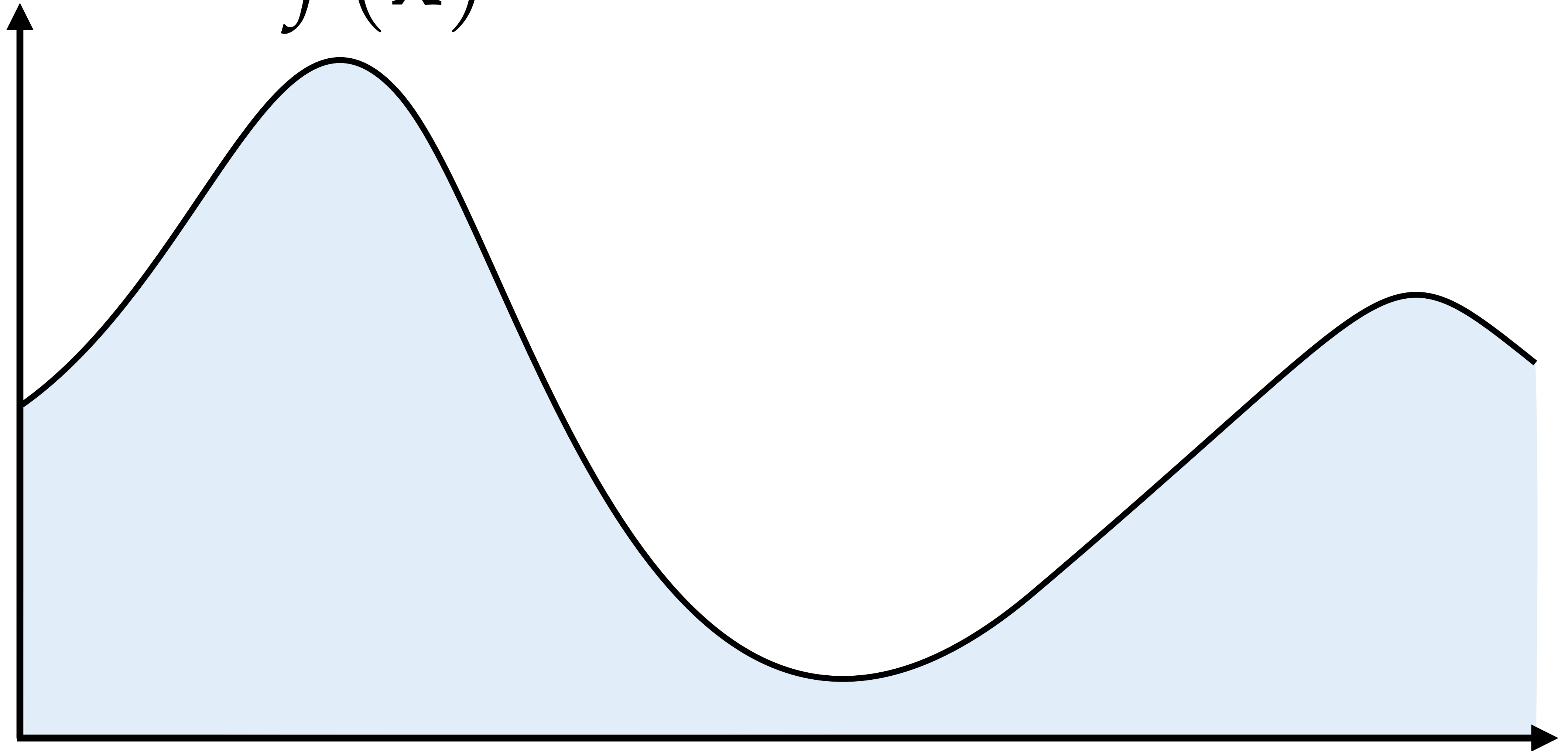
Example 1



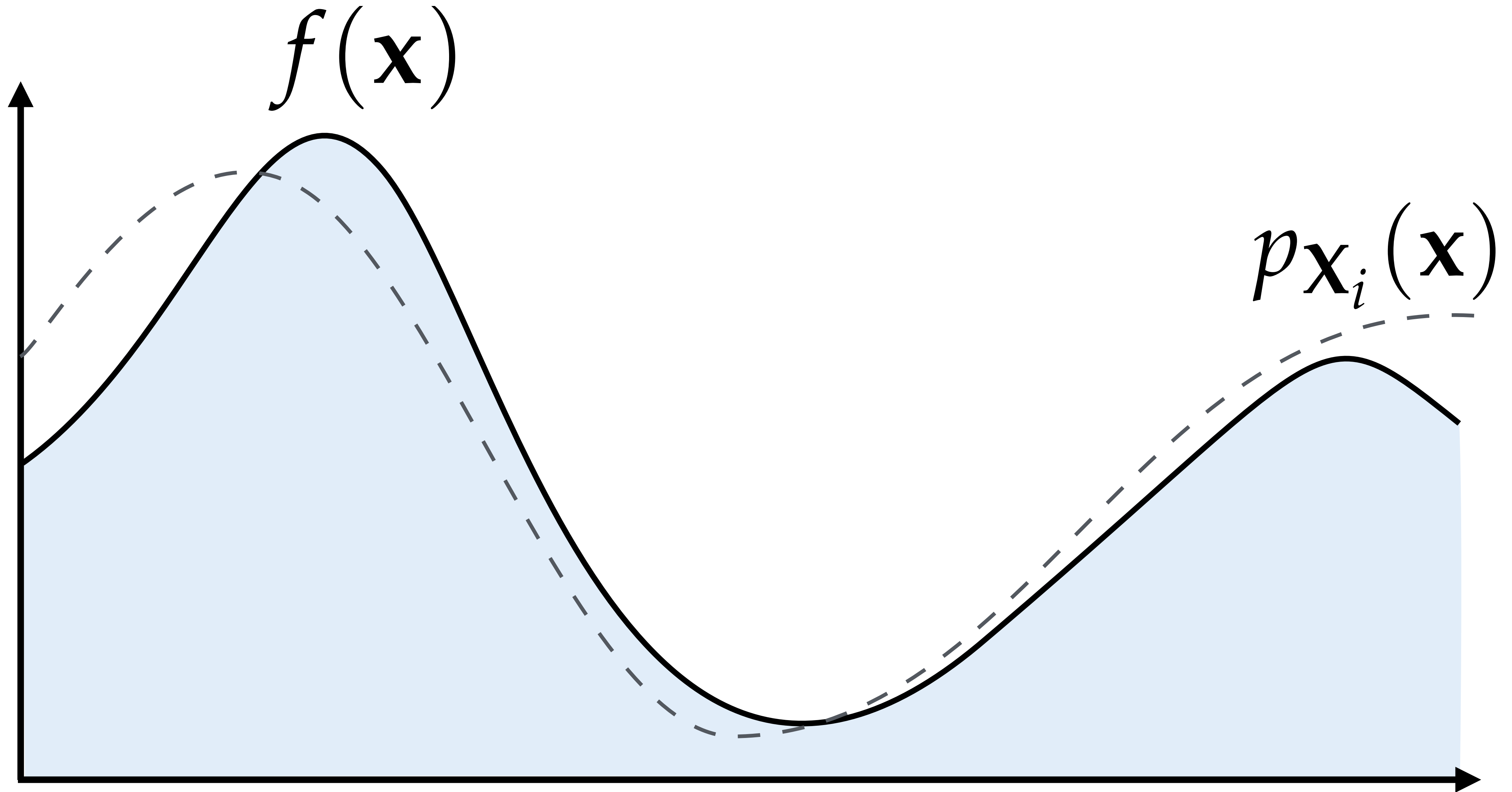
Example 2

# Importance sampling

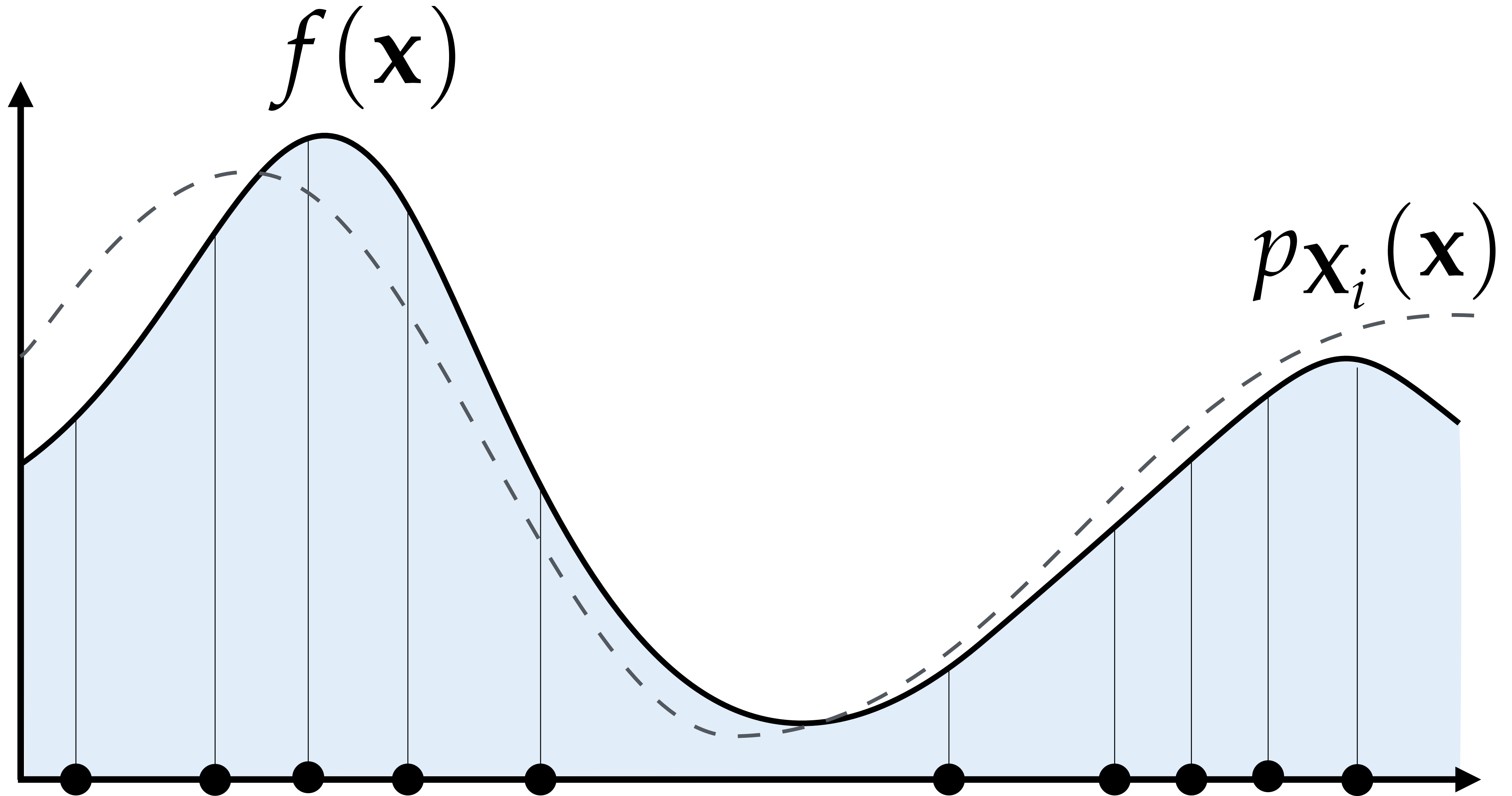
$$f(\mathbf{x})$$



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This is known as **Importance Sampling**.

*Requirement:*  $p_{\mathbf{X}}(\mathbf{x}) > 0$  for all  $\mathbf{x}$  with  $f(\mathbf{x}) \neq 0$ .

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- CS440, my master-level course about graphics will teach you all about this.

# Monte Carlo methods in the film industry

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**Next lecture:  
Inverse Graphics**

(last lecture of the semester)